Heffernan, J. B., D. L. Watts, and M. J. Cohen. Discharge competence as an ecohydrologic mechanism for pattern formation in peatlands: a meta-ecosystem model of the Everglades ridge-slough landscape.

**Supporting Information: Model Analysis S1**

*Local C balance and elevation change:*

Water depth (D) is the difference between surface water level (h) and soil elevation (z):

$D=h-z$ (A1)

So that:

$\frac{dD}{dt}=\dot{D}=\frac{dh}{dt}-\frac{dz}{dt}=\dot{h}-\dot{z}$ (A2)

Local changes in soil elevation are driven by the balance of primary productivity (P) and decomposition (R):

$\frac{dz}{dt}=P-R$ (A3)

We model primary production with water depth as a function of water depth. In this model, gross primary production (P) has a maxima at the long-term mean water depth that is optimum for sawgrass growth (σ), and declines with increasing and decreasing depth:

$P=\left(P\_{s}-P\_{r}\right)\frac{\left(D-σ\right)^{2}}{\left(D-σ\right)^{2}+\left(D\_{T}-σ\right)^{2}}+P\_{r}$ (A4)

where *P*s is the gross peat production in sloughs, *P*r is gross peat production at the optimal depth for sawgrass growth, and *DT* is the depth at which peat accretion is the average of these two end-members (analogous to a half-saturation constant in monod kinetics).

Respiration declines with increasing water depth:

$R=R\_{σ}-r\_{d}\left(D-σ\right)$ (A5)

where *Rσ* is the rate of gross peat decomposition when water levels are at the optimum depth for sawgrass growth, and rd is the rate of respiration decline with depth.

We now define $δ$ as water depth indexed to have a value of 0 when $D=σ$:

$δ=h-z-σ$ (A6)

Substituting Eq. A6 in to expressions for productivity and respiration yields:

$P=\left(P\_{s}-P\_{r}\right)\frac{δ^{2}}{δ^{2}+\left(D\_{T}-σ\right)^{2}}+P\_{r}$ (A7)

and

$=R\_{σ}-r\_{D}δ$ (A8)

Changes in elevation can therefore be described by:

$\frac{dz}{dt}=\left(P\_{s}-P\_{r}\right)\frac{δ^{2}}{δ^{2}+\left(D\_{T}-σ\right)^{2}}+P\_{r}-R\_{σ}+r\_{D}\left(δ\right)$ (A9)

Let

$δ=\hat{δ}∙u\_{D}=h-z-σ=D-σ$ (A10)

And

$δ=\hat{δ}∙u\_{D}=\hat{δ}∙\left(D\_{T}-σ\right)$ (A11)

Substituting Eq. A11 into Eq. A9 yields

$\frac{dz}{dt}=\left(P\_{s}-P\_{r}\right)\frac{\hat{δ}^{2}\left(D\_{T}-σ\right)^{2}}{\hat{δ}^{2}\left(D\_{T}-σ\right)^{2}+\left(D\_{T}-σ\right)^{2}}+P\_{r}-R\_{σ}+r\_{D}\hat{δ}\left(D\_{T}-σ\right)$ (A12)

Which simplifies to:

$\frac{dz}{dt}=\left(P\_{s}-P\_{r}\right)\frac{\hat{δ}^{2}}{\hat{δ}^{2}+1}+P\_{r}-R\_{σ}+r\_{D}\hat{δ}\left(D\_{T}-σ\right)$ (A13)

We now let

$φ=P\_{r}-P\_{s}$ (A14)

$ρ=r\_{D}\left(D\_{T}-σ\right)$ (A15)

So that Eq. A13 can be simplified to:

$\frac{dz}{dt}=P\_{r}-R\_{σ}+ρ\hat{δ}-φ\frac{\hat{δ}^{2}}{\hat{δ}^{2}+1}$ (A16)

Substituting and simplifying yields

$\frac{dD}{dt}=\frac{dh}{dt}-\frac{dz}{dt}=\frac{dh}{dt}-P\_{r}+R\_{σ}-ρ\hat{δ}+φ\frac{\hat{δ}^{2}}{\hat{δ}^{2}+1}$ (A17)

When $D=σ$, the change in elevation is

$\frac{dD}{dt}\_{σ} =\frac{dh}{dt}-P\_{r}+R\_{σ}$ (A18)

So that

$\frac{dD}{dt}=\frac{dD}{dt}\_{σ}-ρ\hat{δ}+φ\frac{\hat{δ}^{2}}{\hat{δ}^{2}+1}$ (A19)

*Lateral coupling*

Local changes in depth resulting from peat accretion and changes in water level are described by:

$\frac{dD\_{i}}{dt}=\frac{dD}{dt}\_{σ}-ρ\hat{δ}\_{i}+φ\frac{\hat{δ}\_{i}^{2}}{\hat{δ}\_{i}^{2}+1}$ (A20)

In our lateral coupling model, water must be routed through the shared cross section of the two adjacent patches, which therefore control water level:

$\frac{dh}{dt}=\frac{1}{2}\left(\frac{dz\_{1}}{dt}+\frac{dz\_{2}}{dt}\right)$ (A21)

Substituting in to Eq. A2 gives changes in depth for each patch in terms of changes in elevation of both patches:

$\frac{dD\_{1}}{dt}=\frac{1}{2}\left(\frac{dz\_{2}}{dt}+\frac{dz\_{1}}{dt}\right)-\frac{dz\_{1}}{dt}=\frac{1}{2}\left(\frac{dz\_{2}}{dt}-\frac{dz\_{1}}{dt}\right)$ (A22)

$\frac{dD\_{2}}{dt}=\frac{1}{2}\left(\frac{dz\_{2}}{dt}+\frac{dz\_{1}}{dt}\right)-\frac{dz\_{2}}{dt}=\frac{1}{2}\left(\frac{dz\_{1}}{dt}-\frac{dz\_{2}}{dt}\right)=-\frac{dD\_{1}}{dt}$ (A23)

Any changes in depth in the two patches are thus equal in magnitude and opposite in sign. Combining Eqs. A20 and A23, we obtain:

$\frac{dD\_{1}}{dt}=\frac{1}{2}\left(\frac{dD}{dt}\_{σ}+ρ\hat{δ}\_{2}-φ\frac{\hat{δ}\_{2}^{2}}{\hat{δ}\_{2}^{2}+1}-\frac{dD}{dt}\_{σ}-ρ\hat{δ}\_{1}+φ\frac{\hat{δ}\_{1}^{2}}{\hat{δ}\_{1}^{2}+1}\right)=0$ (A24)

Depth equilibria in the two patches therefore occur when:

$\frac{dD\_{1}}{dt}=\frac{1}{2}\left(ρ\left(\hat{δ}\_{2}-\hat{δ}\_{1}\right)-φ\left(\frac{\hat{δ}\_{2}^{2}}{\hat{δ}\_{2}^{2}+1}-\frac{\hat{δ}\_{1}^{2}}{\hat{δ}\_{1}^{2}+1}\right)\right)=0$ (A25)

Which is true when:

$\left(\hat{δ}\_{2}-\hat{δ}\_{1}\right)\left(ρ-φ\left(\frac{\hat{δ}\_{1}+\hat{δ}\_{2}}{\left(\hat{δ}\_{2}^{2}+1\right)\left(\hat{δ}\_{1}^{2}+1\right)}\right)\right)=0$ (A26)

The depth in either patch is determined by discharge (*q*), which is a boundary condition, and velocity (*v*), which is assumed to be constant:

$D=\frac{q}{v}$ (A27)

The sum of discharges in the two patches is therefore:

$D\_{1}+D\_{2}=2\frac{q}{v}$ (A28)

When water depths in each patch are indexed to optimum depth for productivity (*σ*), Eq. A28 is equivalent to:

$D\_{1}+D\_{2}-2σ=2\frac{q}{v}-2σ$ (A29)

We now define$ y\_{1,2}$ as the difference in elevation between the two laterally-adjacent patches, so that:

$z\_{2}=z\_{1}+y\_{1,2}$ (A30)

$D\_{2}=D\_{1}-y\_{1,2}$ (A31)

$δ\_{2}=δ\_{1}-y\_{1,2}$ (A32)

Combining Eq. A6 and A29, we have:

$δ\_{1}+δ\_{2}=2\frac{q}{v}-2σ$ (A33)

We now define

$δ\_{i}=\hat{δ}\_{1}∙u\_{D}$ (A34)

So that:

$\hat{δ}\_{1}∙u\_{D}+\hat{δ}\_{2}∙u\_{D}=2\frac{q}{v}-2σ$ (A35)

From which it follows that:

$\hat{δ}\_{1}+\hat{δ}\_{2}=2\frac{q-σv}{v\left(D\_{T}-σ\right)}$ (A36)

Let $\hat{Q}$ be discharge, indexed to sawgrass optimum depth and scaled to $\left(D\_{T}-σ\right):$

$\hat{Q}=\frac{q-σv}{v\left(D\_{T}-σ\right)}$ (A37)

It then follows from Eqs. A36 and A37 that:

$\hat{δ}\_{1}+\hat{δ}\_{2}=2\hat{Q}$ (A38)

We now define $\hat{y}$ as the difference in elevation so that:

$\hat{δ}\_{2}∙u\_{D}=\hat{δ}\_{1}∙u\_{D}-\hat{y}\_{1,2}∙u\_{D}$ (A39)

$\hat{δ}\_{2}=\hat{δ}\_{1}-\hat{y}\_{1,2}$ (A40)

$\hat{δ}\_{1}-\hat{δ}\_{2}=\hat{y}\_{1,2}$ (A41)

Substituting these identities into Eq. A26 yields:

$\hat{y}\_{1,2}\left(φ\left(\frac{2\hat{Q}}{\left(\hat{δ}\_{2}^{2}+1\right)\left(\hat{δ}\_{1}^{2}+1\right)}\right)-ρ\right)=0$ (A42)

Combining Eqs. A38 and A41 yields:

$\hat{δ}\_{1}+\hat{δ}\_{1}-\hat{y}\_{1,2}=2Q$ (A43)

From which we have that:

$\hat{δ}\_{1}=\hat{Q}+\frac{\hat{y}\_{1,2}}{2}$ (A44)

$\hat{δ}\_{2}=\hat{Q}-\frac{\hat{y}\_{1,2}}{2}$ (A45)

We now substitute Eqs. A44 and A45 into Eq. A42 to obtain the final form of the equation describing changes in depth:

$\frac{dD\_{1}}{dt}=\hat{y}\_{1,2}\left(φ\left(\frac{2\hat{Q}}{\left(\left(\hat{Q}+\frac{\hat{y}\_{1,2}}{2}\right)^{2}+1\right)\left(\left(\hat{Q}-\frac{\hat{y}\_{1,2}}{2}\right)^{2}+1\right)}\right)-ρ\right)$ (A46)

*Equilibrium solutions for lateral coupling model*

Depth equilibria occur when

$\frac{dD\_{1}}{dt}=0$ (A47)

Which is equivalent to:

$\hat{y}\_{1,2}\left(φ\left(\frac{2\hat{Q}}{\left(\left(\hat{Q}+\frac{\hat{y}\_{1,2}}{2}\right)^{2}+1\right)\left(\left(\hat{Q}-\frac{\hat{y}\_{1,2}}{2}\right)^{2}+1\right)}\right)-ρ\right)=0$ (A48)

Eq. A48 is true when:

$\hat{y}\_{1,2}=0$ (A49)

Or

$φ\left(\frac{2\hat{Q}}{\left(\frac{\hat{y}\_{1,2}}{2}\right)^{4}+2\left(1-\hat{Q}^{2}\right)\left(\frac{\hat{y}\_{1,2}}{2}\right)^{2}+\left(1+\hat{Q}^{2}\right)^{2}}\right)-ρ=0$ (A50)

Eq. A50 may be expressed as a quadratic in $\left(\frac{\hat{y}\_{1,2}}{2}\right)^{2}$:

$\left(\frac{\hat{y}\_{1,2}}{2}\right)^{4}+2\left(1-\hat{Q}^{2}\right)\left(\frac{\hat{y}\_{1,2}}{2}\right)^{2}+\left(1+\hat{Q}^{2}\right)^{2}-2\hat{Q}\left(\frac{φ}{ρ}\right)=0$ (A51)

Which has solutions:

$\left(\frac{\hat{y}\_{1,2}}{2}\right)^{2}=\frac{-b\pm \sqrt{b^{2}-4ac}}{2a}$ (A52)

where

$a=1$ (A53)

$b=2\left(1-\hat{Q}^{2}\right)$ (A54)

$c=\left(1+\hat{Q}^{2}\right)^{2}-2\hat{Q}\left(\frac{φ}{ρ}\right)$ (A55)

Since

$b^{2}=4\left(1-\hat{Q}^{2}\right)^{2}$ (A56)

and

$-4ac=8\hat{Q}\left(\frac{φ}{ρ}\right)-4\left(1+\hat{Q}^{2}\right)^{2}$ (A57)

It follows that:

$b^{2}-4ac=4\left(1-\hat{Q}^{2}\right)^{2}+8\hat{Q}\left(\frac{φ}{ρ}\right)-4\left(1+\hat{Q}^{2}\right)^{2}=$ (A58)

Which can simplified to:

$b^{2}-4ac=8\hat{Q}\left(\frac{φ}{ρ}\right)-16\hat{Q}^{2}=$ (A59)

$b^{2}-4ac=16\hat{Q}\left(\frac{φ}{2ρ}-\hat{Q}\right)$ (A60)

Therefore solutions to Eq. A48 are:

$\left(\frac{\hat{y}\_{1,2}}{2}\right)^{2}=\frac{-2\left(1-\hat{Q}^{2}\right)\pm 4\sqrt{\hat{Q}\left(\frac{φ}{2ρ}-\hat{Q}\right)}}{2}$ (A61)

Since the designation of patches 1 and 2 are entirely arbitrary, we take the positive and negative roots for values of $\hat{y}\_{1,2}$ solutions to be equivalent solutions. Equilibrium depths for laterally-adjacent patches therefore occur when:

$\hat{y}\_{1,2}=-2\sqrt{\hat{Q}^{2}-1\pm 2\sqrt{\hat{Q}\left(\frac{φ}{2ρ}-\hat{Q}\right)}}$ (A62)

*Bounding existence of solutions to lateral coupling model*

The trivial solution to Eq. A46 at $\hat{y}\_{1,2}=0$ always exists.

Solutions at

$\hat{y}\_{1,2}=-2\sqrt{\hat{Q}^{2}-1\pm 2\sqrt{\hat{Q}\left(\frac{φ}{2ρ}-\hat{Q}\right)}}$ (A63)

Exist only if

$\hat{Q}\left(\frac{φ}{2ρ}-\hat{Q}\right)>0$ (A64)

Which requires

$0<\hat{Q}<\frac{φ}{2ρ}$ (A65)

Assuming $\frac{φ}{2ρ}>0$, there is always some Q for which this is true.

The solution at

$\hat{y}\_{1,2}=-2\sqrt{\hat{Q}^{2}-1+2\sqrt{\hat{Q}\left(\frac{φ}{2ρ}-\hat{Q}\right)}}$ (A66)

Further requires that

$\hat{Q}^{2}-1+2\sqrt{\hat{Q}\left(\frac{φ}{2ρ}-\hat{Q}\right)}>0$ (A67)

$1-\hat{Q}^{2}<2\sqrt{\hat{Q}\left(\frac{φ}{2ρ}-\hat{Q}\right)}$ (A68)

This is true for $1<\hat{Q}<\frac{φ}{2ρ}$, over which range a real solution to Eq. A66 always exists.

For $\hat{Q}<1$, a real solution to Eq. A66 exists when:

$1-\hat{Q}^{2}<2\sqrt{\hat{Q}\left(\frac{φ}{2ρ}-\hat{Q}\right)}$ (A69)

Which is equivalent to:

$\hat{Q}^{4}+2\hat{Q}^{2}-2\hat{Q}\frac{φ}{ρ}+1<0$ (A70)

Since the value of Eq. A70 at the limit is positive, a real solution only exists for $\hat{Q}<1$ if

$\hat{Q}^{4}+2\hat{Q}^{2}-2\hat{Q}\frac{φ}{ρ}+1=0$ (A71)

Has real solutions. The discriminant for a quartic function ($D\_{4}) $is:

$$D\_{4}=\left(a\_{1}^{2}a\_{2}^{2}a\_{3}^{2}-4a\_{1}^{3}a\_{3}^{3}-4a\_{1}^{2}a\_{2}^{3}a\_{4}^{}+18a\_{1}^{3}a\_{2}^{}a\_{3}^{}a\_{4}^{}-27a\_{1}^{4}a\_{4}^{2}+256a\_{0}^{3}a\_{4}^{3}\right)+a\_{0}^{}\left(-4a\_{2}^{3}a\_{3}^{2}+18a\_{1}^{}a\_{2}^{}a\_{3}^{3}+16a\_{2}^{4}a\_{4}^{}-80a\_{1}^{}a\_{2}^{2}a\_{3}^{}a\_{4}^{}-6a\_{1}^{2}a\_{3}^{2}a\_{4}^{}+144a\_{1}^{2}a\_{2}^{}a\_{4}^{2}\right)+a\_{0}^{2}\left(-27a\_{3}^{4}+144a\_{2}^{}a\_{3}^{2}a\_{4}^{}-128a\_{2}^{2}a\_{4}^{2}-192a\_{1}^{}a\_{3}^{}a\_{4}^{2}\right)$$

(A72)

Eq. A71 has real solutions if $D\_{4}<0$. For Eq. A71, parameters of the discriminant are:

$a\_{0}^{}=1$ (A73)

$a\_{1}^{}=-2\frac{φ}{ρ}$ (A74)

$a\_{2}^{}=2$ (A75)

$a\_{3}^{}=0$ (A76)

$a\_{4}^{}=1$ (A77)

Substituting these parameters in to the generic determinant formula yields:

$D\_{4}=-32a\_{1}^{2}-27a\_{1}^{4}+256+256+288a\_{1}^{2}-512$ (A78)

Which simplifies to:

$D\_{4}=256a\_{1}^{2}-27a\_{1}^{4}$ (A79)

$D\_{4}=a\_{1}^{2}\left(256-108\left(\frac{φ}{ρ}\right)^{2}\right)$ (A80)

Eq. A71 has real solutions when:

$a\_{1}^{2}\left(256-108\left(\frac{φ}{ρ}\right)^{2}\right)<0$ (A81)

Which is true only when:

$\frac{φ}{ρ}>\frac{8}{3\sqrt{3}}$ (A82)

The actual value of the solutions to Eq. A70, which determine the discharge at which bifurcations occur, are given by Eqs. A110-112. We present analyses of these solutions after some preliminary analysis of the other nonzero solution.

The solution at

$\hat{y}\_{1,2}=-2\sqrt{\hat{Q}^{2}-1-2\sqrt{\hat{Q}\left(\frac{φ}{2ρ}-\hat{Q}\right)}}$ (A83)

Is real if

$\hat{Q}^{2}-1>2\sqrt{\hat{Q}\left(\frac{φ}{2ρ}-\hat{Q}\right)}$ (A84)

Which is never true for $\hat{Q}<1$. Since $0<\hat{Q}<\frac{φ}{2ρ}$, it follows that a real solution exists when $\frac{φ}{ρ}>2$.

Assuming $\hat{Q}>1$, Eq. A83 has a real solution when

$\hat{Q}^{2}-1>2\sqrt{\hat{Q}\left(\frac{φ}{2ρ}-\hat{Q}\right)}$ (A85)

Which is equivalent to:

$\hat{Q}^{4}+2\hat{Q}^{2}-2\hat{Q}\frac{φ}{ρ}+1>0$ (A86)

Given the generic quartic equation:

$Ax^{4}+Bx^{3}+Cx^{2}+Dx+E$ (A87)

its solution can be found by means of the following calculations (Cardano’s solution). Let:

$α=-\frac{3B^{2}}{8A^{2}}+\frac{C}{A}$ (A88)

$β=\frac{B^{3}}{8A^{3}}-\frac{BC}{2A^{2}}+\frac{D}{A}$ (A89)

$γ=-\frac{3B^{4}}{256A^{4}}+\frac{CB^{2}}{16A^{3}}-\frac{BD}{24}+\frac{E}{A}$ (A90)

For Eq. A86,

$α=2$ (A91)

$β=-\frac{2φ}{ρ}$ (A92)

$γ=1$ (A93)

We now let:

$P=-\frac{α^{2}}{12}-γ$ (A94)

$Q=-\frac{α^{3}}{108}+\frac{αγ}{3}-\frac{β^{3}}{8}$ (A95)

$R=-\frac{Q}{2}\pm \sqrt{\frac{Q^{2}}{4}+\frac{P^{3}}{27}}$ (A96)

For Eq. A86,

$Q=-\frac{8}{108}+\frac{2}{3}-\frac{1}{2}\left(\frac{φ}{ρ}\right)^{2}=\frac{16}{27}-\frac{1}{2}\left(\frac{φ}{ρ}\right)^{2}$ (A97)

$P=-\frac{4}{12}-1=-\frac{4}{3}$ (A98)

$R=\left(\frac{φ}{2ρ}\right)^{2}-\frac{8}{27}\pm \left(\frac{φ}{2ρ}\right)\sqrt{\left(\frac{φ}{2ρ}\right)^{2}-\frac{16}{27}}$ (A99)

We now let

$U=\sqrt[3]{R}=\sqrt[3]{\left(\frac{φ}{2ρ}\right)^{2}-\frac{8}{27}+\left(\frac{φ}{2ρ}\right)\sqrt{\left(\frac{φ}{2ρ}\right)^{2}-\frac{16}{27}}}$ (A100)

And let

$y=-\frac{5}{6}α+U-\frac{P}{3U}=-\frac{5}{3}+U+\frac{4}{9U}$ (A101)

So that

$y=\frac{9U^{2}-15U+4}{9U}$ (A102)

We further let

$W=\sqrt{α+2y}$ (A103)

So that

$W=\sqrt{2+2\left(-\frac{5}{3}+U+\frac{4}{9U}\right)}=\sqrt{-\frac{4}{3}+2U+\frac{8}{9U}}=\sqrt{\frac{18U^{2}-12U+8}{9U}}$ (A104)

Solutions to the quartic equation are given by:

$x=-\frac{B}{4a}+\frac{\pm \_{s} W \mp \_{t}\sqrt{-(3α+2y\pm \frac{2β}{W}}}{2}$ (A105)

For Eq. A86, the root

$\sqrt{-(3α+2y\pm \frac{2β}{W}}$ (A106)

Is equivalent to

$\sqrt{-\left(6+2\left(-\frac{5}{3}+U+\frac{4}{9U}\right)+\frac{2β}{\sqrt{\frac{18U^{2}-12U+8}{9U}}}\right)}=\sqrt{-\left(3α-\frac{5}{3}+2U-\frac{8}{9U}+\frac{2β}{W}\right)}$ (A107)

$=\sqrt{-\left(\frac{4}{3}α+2U-\frac{2P}{3U}+\frac{2β}{W}\right)}$ (A108)

$=\sqrt{4\frac{φ}{ρ}\frac{1}{W}-4-W^{2}}$ (A109)

Solutions to Eq. A86 occur at

$Q=\frac{W}{2}\pm \sqrt{4\frac{φ}{ρ}\frac{1}{W}-4-W^{2}}$ (A110)

Where

$W=\sqrt{\frac{18U^{2}-12U+8}{9U}}$ (A111)

And

$U=\sqrt[3]{\left(\frac{φ}{2ρ}\right)^{2}-\frac{8}{27}+\left(\frac{φ}{2ρ}\right)\sqrt{\left(\frac{φ}{2ρ}\right)^{2}-\frac{16}{27}}}$ (A112)

**Summary of bifurcations**

The upper bound of both equilibria is $\hat{Q}<\frac{φ}{2ρ}$.

The lower bound of the equilibrium depth difference

$\hat{y}\_{1,2}=-2\sqrt{\hat{Q}^{2}-1-2\sqrt{\hat{Q}\left(\frac{φ}{2ρ}-\hat{Q}\right)}}$ (A113)

Requires $\hat{Q}>1$, and so occurs when

$Q=\frac{W}{2}+\sqrt{4\frac{φ}{ρ}\frac{1}{W}-4-W^{2}}$ (A114)

This solution exists only if $\frac{φ}{ρ}>2.$

The lower bound of the equilibrium depth difference

$\hat{y}\_{1,2}=-2\sqrt{\hat{Q}^{2}-1+2\sqrt{\hat{Q}\left(\frac{φ}{2ρ}-\hat{Q}\right)}}$ (A115)

Requires $\hat{Q}<1$, and so occurs when

$Q=\frac{W}{2}-\sqrt{4\frac{φ}{ρ}\frac{1}{W}-4-W^{2}}$ (A116)

This solution exists only when $\frac{φ}{ρ}>\frac{8}{3\sqrt{3}}$

*Stability of solutions to lateral coupling model*

To assess the stability of equilibrium depth differences, we take the derivative of Eq. A46 with respect to $\hat{y}\_{1,2}$, which is:

$\frac{d\dot{D}\_{1}}{d\hat{y}\_{1,2}}=\frac{2φ\hat{Q}\left(-3\left(\frac{\hat{y}\_{1,2}}{2}\right)^{4}-3\left(1-\hat{Q}^{2}\right)\left(\frac{\hat{y}\_{1,2}}{2}\right)^{2}+\left(1+\hat{Q}^{2}\right)^{2}\right)}{\left(\left(\frac{\hat{y}\_{1,2}}{2}\right)^{4}+2\left(1-\hat{Q}^{2}\right)\left(\frac{\hat{y}\_{1,2}}{2}\right)^{2}+\left(1+\hat{Q}^{2}\right)^{2}\right)^{2}}-ρ$ (A117)

Evaluating this derivative for $\hat{y}\_{1,2}=0$ yields:

$\frac{d\dot{D}\_{1}}{d\hat{y}\_{1,2}}\_{y1,2=0}=\frac{2φ\hat{Q}}{\left(1+\hat{Q}^{2}\right)^{2}}-ρ$ (A118)

The equilibrium is stable when

$\frac{2φ\hat{Q}}{\left(1+\hat{Q}^{2}\right)^{2}}-ρ<0$ (A119)

Which is equivalent to

$\left(1+\hat{Q}^{2}\right)^{2}-2\hat{Q}\frac{φ}{ρ}>0$ (A120)

So the equilibrium at $\hat{y}\_{1,2}=0$ is unstable between the lower bounds (bifurcations) for the other two equilibria.

The derivative of Eq. A46 can also be expressed as:

$\frac{d\dot{D}\_{1}}{d\hat{y}\_{1,2}}=\frac{\left(\left(\frac{\hat{y}\_{1,2}}{2}\right)^{4}+2\left(1-\hat{Q}^{2}\right)\left(\frac{\hat{y}\_{1,2}}{2}\right)^{2}+\left(1+\hat{Q}^{2}\right)^{2}\right)2φ\hat{Q}-2φ\hat{Q}\hat{y}\_{1,2}\left(2\left(\frac{\hat{y}\_{1,2}}{2}\right)^{3}+\left(1-\hat{Q}^{2}\right)\hat{y}\_{1,2}\right)}{\left(\left(\frac{\hat{y}\_{1,2}}{2}\right)^{4}+2\left(1-\hat{Q}^{2}\right)\left(\frac{\hat{y}\_{1,2}}{2}\right)^{2}+\left(1+\hat{Q}^{2}\right)^{2}\right)^{2}}-ρ$ (A121)

Eq. A51 can be re-arranged as:

$\left(\frac{\hat{y}\_{1,2}}{2}\right)^{4}+2\left(1-\hat{Q}^{2}\right)\left(\frac{\hat{y}\_{1,2}}{2}\right)^{2}+\left(1+\hat{Q}^{2}\right)^{2}=2\hat{Q}\left(\frac{φ}{ρ}\right)$ (A122)

Which we substitute in to Eq. A121:

$\frac{d\dot{D}\_{1}}{d\hat{y}\_{1,2}}=\frac{\left(\frac{2φ\hat{Q}}{ρ}\right)2φ\hat{Q}-2φ\hat{Q}\hat{y}\_{1,2}\left(2\left(\frac{\hat{y}\_{1,2}}{2}\right)^{3}+\left(1-\hat{Q}^{2}\right)\hat{y}\_{1,2}\right)}{\left(\frac{2φ\hat{Q}}{ρ}\right)^{2}}-ρ$ (A123)

This expression simplifies to:

$\frac{d\dot{D}\_{1}}{d\hat{y}\_{1,2}}=-\frac{ρ^{2}\left(\hat{y}\_{1,2}\right)^{2}\left(\left(\frac{\hat{y}\_{1,2}}{2}\right)^{2}+\left(1-\hat{Q}^{2}\right)\right)}{2φ\hat{Q}}$ (A124)

Evaluating for the equilibrium given by Eq. A62, we have that

$\left(\frac{\hat{y}\_{1,2}}{2}\right)^{2}=\hat{Q}^{2}-1+2\sqrt{\hat{Q}\left(\frac{φ}{2ρ}-\hat{Q}\right)}$ (A125)

Which we substitute into Eq. A124 to obtain:

$\frac{d\dot{D}\_{1}}{d\hat{y}\_{1,2}}=-\frac{ρ^{2}\left(\hat{y}\_{1,2}\right)^{2}\left(2\sqrt{\hat{Q}\left(\frac{φ}{2ρ}-\hat{Q}\right)}\right)}{2φ\hat{Q}}$ (A126)

Since both the numerator and the denominator are always positive, the equilibrium at

$\hat{y}\_{1,2}=-2\sqrt{\hat{Q}^{2}-1+2\sqrt{\hat{Q}\left(\frac{φ}{2ρ}-\hat{Q}\right)}}$ (A127)

Is always stable when it exists.

Evaluating for the equilibrium given by Eq. A62, we have that

$\left(\frac{\hat{y}\_{1,2}}{2}\right)^{2}=\hat{Q}^{2}-1-2\sqrt{\hat{Q}\left(\frac{φ}{2ρ}-\hat{Q}\right)}$ (A128)

Which we substitute into Eq. A124 to obtain:

$\frac{d\dot{D}\_{1}}{d\hat{y}\_{1,2}}=\frac{ρ^{2}\left(\hat{y}\_{1,2}\right)^{2}\left(2\sqrt{\hat{Q}\left(\frac{φ}{2ρ}-\hat{Q}\right)}\right)}{2φ\hat{Q}}$ (A129)

Since the numerator and denominator are always positive, the equilibrium at

$\hat{y}\_{1,2}=-2\sqrt{\hat{Q}^{2}-1-2\sqrt{\hat{Q}\left(\frac{φ}{2ρ}-\hat{Q}\right)}}$ (A130)

Is always unstable when it exists.

*Longitudinal coupling*

We now consider a third patch, with soil elevaton $z\_{3}$, located downstream of the deeper of two patches adjacent upstream patches (i.e. $D\_{1}$ and $D\_{2})$. Further, we assume that $D\_{1}$ and $D\_{2}$ at equilibrium, meaning that

$\hat{y}\_{1,2}=-2\sqrt{\hat{Q}^{2}-1+2\sqrt{\hat{Q}\left(\frac{φ}{2ρ}-\hat{Q}\right)}}$ (A131)

or

$\hat{y}\_{1,2}=0$ (A132)

Since the equilibrium at

$\hat{y}\_{1,2}=-2\sqrt{\hat{Q}^{2}-1-2\sqrt{\hat{Q}\left(\frac{φ}{2ρ}-\hat{Q}\right)}}$ (A133)

Is unstable and so unlikely to persist as a constraint on water levels to patch $D\_{3}$. We take as given that $D\_{1}$is the higher elevation (shallower depth) of the two upstream patches.

Changes in depth are governed by the same carbon balance responses to water depth as the upstream patches

$\frac{dD\_{3}}{dt}=\frac{dh}{dt}-\frac{dz\_{3}}{dt}=\frac{dh}{dt}-\dot{z}\_{0}+ρ\hat{δ}\_{3}-φ\frac{\hat{δ}\_{3}^{2}}{\hat{δ}\_{3}^{2}+1}$ (A134)

However, we assume that $z\_{3}\leq z\_{1}$ so that $z\_{3} $does not control but only responds to water levels.

Changes in water level for the downstream patch ($\hat{δ}\_{3}$) are dependent on the changes in elevation (and thus water level) of the two upstream patches:

$\frac{dh}{dt}=\frac{1}{2}\left(\frac{dz\_{1}}{dt}+\frac{dz\_{2}}{dt}\right)=\frac{dz\_{1}}{dt}$ (A135)

So that changes in depth for the downstream patch are governed by:

$\frac{dD\_{3}}{dt}=\frac{dD}{dt}\_{σ}-ρ\hat{δ}\_{1}+φ\frac{\hat{δ}\_{1}^{2}}{\hat{δ}\_{1}^{2}+1}-\frac{dD}{dt}\_{σ}+ρ\hat{δ}\_{3}-φ\frac{\hat{δ}\_{3}^{2}}{\hat{δ}\_{3}^{2}+1}$ (A136)

Letting

$z\_{3}=z\_{1}+y\_{1,3}$ (A137)

$D\_{3}=D\_{1}-y\_{1,3}$ (A138)

$δ\_{3}=δ\_{1}-y\_{1,3}$ (A139)

$\hat{y}\_{1,3}=\hat{δ}\_{1}-\hat{δ}\_{3}$ (A140)

Eq. A136 can be simplified to:

$\frac{dD\_{3}}{dt}=\hat{y}\_{1,3}\left(φ\frac{2\hat{δ}\_{1}-\hat{y}\_{1,3}}{\left(\hat{δ}\_{1}^{2}+1\right)\left(\left(\hat{δ}\_{1}-\hat{y}\_{1,3}\right)^{2}+1\right)}-ρ\right)$ (A141)

Equilibria occur where

$\hat{y}\_{1,3}\left(φ\frac{2\hat{δ}\_{1}-\hat{y}\_{1,3}}{\left(\hat{δ}\_{1}^{2}+1\right)\left(\left(\hat{δ}\_{1}-\hat{y}\_{1,3}\right)^{2}+1\right)}-ρ\right)=0$ (A142)

The trivial solution ($\hat{y}\_{1,3}=0)$ is equivalent to:

$∴\hat{δ}\_{3}=\hat{δ}\_{1}$ (A143)

Additional solutions occur when:

$φ\frac{2\hat{δ}\_{1}-\hat{y}\_{1,3}}{\left(\hat{δ}\_{1}^{2}+1\right)\left(\left(\hat{δ}\_{1}-\hat{y}\_{1,3}\right)^{2}+1\right)}-ρ=0$ (A144)

Since $D\_{3}$ follows same rules as $D\_{1}$, it seems intuitive that an equilibrium would also occur when

$∴\hat{δ}\_{3}=\hat{δ}\_{2}$ (A145)

From which it follows that

$\hat{δ}\_{3}=\hat{δ}\_{1}-\hat{y}\_{1,3}=\hat{δ}\_{2}=\hat{δ}\_{1}-\hat{y}\_{1,2}$ (A146)

and

$\hat{y}\_{1,3}=\hat{y}\_{1,2}$ (A147)

So that Eq. A141 would be equivalent to:

$φ\frac{2\hat{Q}}{\left(\left(\hat{Q}+\frac{\hat{y}\_{1,3}}{2}\right)^{2}+1\right)\left(\left(\hat{Q}-\frac{\hat{y}\_{1,3}}{2}\right)^{2}+1\right)}-ρ=0$ (A148)

For which

$\hat{y}\_{1,3}=-2\sqrt{\hat{Q}^{2}-1+2\sqrt{\hat{Q}\left(\frac{φ}{2ρ}-\hat{Q}\right)}}$ (A149)

has already been shown to be a solution (Eqs. A47-A62), so this is also a solution for $\hat{y}\_{1,3}$.

$φ\frac{2\hat{δ}\_{1}-\hat{y}\_{1,3}}{\left(\hat{δ}\_{1}^{2}+1\right)\left(\left(\hat{δ}\_{1}-\hat{y}\_{1,3}\right)^{2}+1\right)}-ρ=0$ (A150)

Can be expressed as:

$\hat{y}\_{1,3}^{2}+\hat{y}\_{1,3}\left(\frac{φ}{ρ\left(\hat{δ}\_{1}^{2}+1\right)}-2\hat{δ}\_{1}\right)+\left(\hat{δ}\_{1}^{2}+1\right)-\frac{2\hat{δ}\_{1}φ}{ρ\left(\hat{δ}\_{1}^{2}+1\right)}=0$ (A151)

Which is a quadratic in $y\_{1,3}$. So there remains one additional root, which we will obtain by factoring out the preceding solution. First, we selectively substitute for:

$\hat{δ}\_{1}=\hat{Q}+\frac{\hat{y}\_{1,2}}{2}$ (A152)

Which leaves us

$\hat{y}\_{1,3}^{2}+\hat{y}\_{1,3}\left(\frac{φ}{ρ\left(\hat{δ}\_{1}^{2}+1\right)}-2\hat{Q}-\hat{y}\_{1,2}\right)+\left(\hat{δ}\_{1}^{2}+1\right)-\frac{2\hat{δ}\_{1}φ}{ρ\left(\hat{δ}\_{1}^{2}+1\right)}=0$ (A153)

Let A be such that:

$\left(\hat{y}\_{1,3}-\hat{y}\_{1,2}\right)\left(\hat{y}\_{1,3}-A\right)=\hat{y}\_{1,3}^{2}+\hat{y}\_{1,3}\left(\frac{φ}{ρ\left(\hat{δ}\_{1}^{2}+1\right)}-2\hat{Q}-\hat{y}\_{1,2}\right)+\left(\hat{δ}\_{1}^{2}+1\right)-\frac{2\hat{δ}\_{1}φ}{ρ\left(\hat{δ}\_{1}^{2}+1\right)}$ (A154)

$-A-\hat{y}\_{1,2}=\frac{φ}{ρ\left(\hat{δ}\_{1}^{2}+1\right)}-2\hat{Q}-\hat{y}\_{1,2}$ (A155)

$A=2\hat{Q}-\frac{φ}{ρ\left(\hat{δ}\_{1}^{2}+1\right)}$ (A156)

Therefore

$\hat{y}\_{1,3}=2\hat{Q}-\frac{φ}{ρ\left(\hat{δ}\_{1}^{2}+1\right)}$ (A157)

Is the third solution to Eq. A141.

*Assessing stability of longitudinally-coupled patches*

To assess the stability of equilibrium depth differences, we take the derivative of Eq. A141, which is:

$\frac{dD\_{3}}{dy\_{1,3}}=φ\frac{2\left(\hat{δ}\_{1}-\hat{y}\_{1,3}\right)}{\left(\left(\hat{δ}\_{1}-\hat{y}\_{1,3}\right)^{2}+1\right)^{2}}-ρ$ (A158)

Equilibria are stable when:

$\frac{dD\_{3}}{d\hat{y}\_{1,3}}<0$ (A159)

When $y\_{1,3}=0, $

$\frac{dD\_{3}}{d\hat{y}\_{1,3}}\_{y\_{1,3}=0}=φ\frac{2\hat{δ}\_{1}}{\left(\hat{δ}\_{1}^{2}+1\right)^{2}}-ρ$ (A160)

Given

$A=2\hat{Q}-\frac{φ}{ρ\left(\hat{δ}\_{1}^{2}+1\right)}$ (A161)

It follows that

$\frac{2φ\hat{δ}\_{1}}{ρ}-\left(\hat{δ}\_{1}^{2}+1\right)^{2}=\frac{φ}{ρ}\hat{y}\_{1,2}-2\hat{Q}\hat{y}\_{1,2}\left(\hat{δ}\_{1}^{2}+1\right)=-A\hat{y}\_{1,2}\left(\hat{δ}\_{1}^{2}+1\right)$ (A162)

From which we have that

$\frac{dD\_{3}}{dy\_{1,3}}\_{y\_{1,3}=0}=φ\frac{2\hat{δ}\_{1}}{\left(\hat{δ}\_{1}^{2}+1\right)^{2}}-ρ=\hat{y}\_{1,2}\frac{-ρA}{\left(\hat{δ}\_{1}^{2}+1\right)^{2}}$ (A163)

And is therefore negative when $A>0$. The equilibrium at $\hat{y}\_{1,3}=0$ is stable whenever $\hat{δ}\_{1}<\hat{δ}\_{3}$, and therefore whenever it exists.

To assess stability of other equilibria, we first expand Eq. A141:

$-\frac{2φy\_{1,3}}{ρ}+\frac{2φ\hat{δ}\_{1}}{ρ}-\left(\hat{δ}\_{1}^{2}+1\right)^{2}+2\left(\hat{δ}\_{1}^{2}+1\right)\left(2\hat{δ}\_{1}y\_{1,3}-y\_{1,3}^{2}\right)-\left(2\hat{δ}\_{1}y\_{1,3}-y\_{1,3}^{2}\right)^{2}<0$

(A164)

And substitute using Eq. A156

$-Ay\left(\hat{δ}\_{1}^{2}+1\right)-\frac{2φy\_{1,3}}{ρ}+2\left(\hat{δ}\_{1}^{2}+1\right)\left(2\hat{δ}\_{1}y\_{1,3}-y\_{1,3}^{2}\right)-\left(2\hat{δ}\_{1}y\_{1,3}-y\_{1,3}^{2}\right)^{2}<0$ (A165)

When $\hat{y}\_{1,3}=\hat{y}\_{1,2}$,

$\frac{d\dot{D}\_{3}}{dy\_{1,3}}\_{y\_{1,3}=\hat{y}\_{1,2}}=-A\hat{y}\_{1,2}\left(\hat{δ}\_{1}^{2}+1\right)-\frac{2φ\hat{y}\_{1,2}}{ρ}+2\left(\hat{δ}\_{1}^{2}+1\right)\left(2\hat{Q}\hat{y}\_{1,2}\right)-\left(2\hat{Q}\hat{y}\_{1,2}\right)^{2}$ (A166)

Substituting based on Eq. A156, we have that:

$\frac{d\dot{D}\_{3}}{dy\_{1,3}}\_{y\_{1,3}=\hat{y}\_{1,2}}=A\hat{y}\_{1,2}\left(\hat{δ}\_{1}^{2}+1\right)-\left(2\hat{Q}\hat{y}\_{1,2}\right)^{2}<0$ (A167)

Which is true when:

$A\left(\hat{δ}\_{1}^{2}+1\right)>4\hat{Q}^{2}\hat{y}\_{1,2}$ (A168)

To evaluate this identity, we begin with

$A=2\hat{Q}-\frac{φ}{ρ\left(\hat{δ}\_{1}^{2}+1\right)}$ (A169)

At equilibrium for $\hat{y}\_{1,2}$

$φ\left(\frac{2\hat{Q}}{\left(\hat{δ}\_{2}^{2}+1\right)\left(\hat{δ}\_{1}^{2}+1\right)}\right)-ρ=0$ (A170)

Which can also be expressed as

$\frac{φ}{ρ}=\frac{\left(\hat{δ}\_{2}^{2}+1\right)\left(\hat{δ}\_{1}^{2}+1\right)}{2\hat{Q}}$ (A171)

Eq. A156 is therefore equivalent to:

$A=2\hat{Q}-\frac{\left(\hat{δ}\_{2}^{2}+1\right)}{2\hat{Q}}$ (A172)

Since

$\hat{δ}\_{2}^{2}+1=-\hat{Q}\hat{y}\_{1,2}+2\hat{Q}^{2}+2\sqrt{\hat{Q}\left(\frac{φ}{2ρ}-\hat{Q}\right)}$ (A173)

Eq. A156 is also equivalent to:

$A=2\hat{Q}-\frac{-\hat{Q}\hat{y}\_{1,2}+2\hat{Q}^{2}+2\sqrt{\hat{Q}\left(\frac{φ}{2ρ}-\hat{Q}\right)}}{2\hat{Q}}$ (A174)

and

$A=\hat{Q}+\frac{\hat{y}\_{1,2}}{2}-\frac{\sqrt{\hat{Q}\left(\frac{φ}{2ρ}-\hat{Q}\right)}}{\hat{Q}}$ (A175)

If we assume

$A>\hat{y}\_{1,2}$ (A176)

Then it follows that

$\hat{Q}-\frac{\sqrt{\hat{Q}\left(\frac{φ}{2ρ}-\hat{Q}\right)}}{\hat{Q}}<\frac{\hat{y}\_{1,2}}{2}$ (A177)

$2\hat{Q}^{2}<\hat{Q}\hat{y}\_{1,2}+2\sqrt{\hat{Q}\left(\frac{φ}{2ρ}-\hat{Q}\right)}$ (A178)

$4\hat{Q}^{2}<\hat{Q}\hat{y}\_{1,2}+2\hat{Q}^{2}+2\sqrt{\hat{Q}\left(\frac{φ}{2ρ}-\hat{Q}\right)}$ (A179)

and therefore that

$\hat{δ}\_{1}^{2}+1>4Q^{2}$ (A180)

Since $\hat{y}\_{1,2}<0, $

$\hat{y}\_{1,2}\left(A\left(\hat{δ}\_{1}^{2}+1\right)-4Q^{2}\hat{y}\_{1,2}\right)<0$ (A181)

Will be true whenever $\hat{y}\_{1,2}<A$. The equilibrium where $\hat{y}\_{1,3}=\hat{y}\_{1,2}$ will therefore be stable when

$\hat{y}\_{1,3}<A$, in which case it is the deepest of the three equilibria.

To assess stability of the third equilibrium ($\hat{y}\_{1,3}=A)$, we again begin by expanding Eq. A141 to:

$-\frac{2φy\_{1,3}}{ρ}+\frac{2φ\hat{δ}\_{1}}{ρ}-\left(\hat{δ}\_{1}^{2}+1\right)^{2}+2\left(\hat{δ}\_{1}^{2}+1\right)\left(2\hat{δ}\_{1}y\_{1,3}-y\_{1,3}^{2}\right)-\left(2\hat{δ}\_{1}y\_{1,3}-y\_{1,3}^{2}\right)^{2}<0$

(A182)

And substitute in based on Eq. A156.

$-A\hat{y}\_{1,2}\left(\hat{δ}\_{1}^{2}+1\right)+\frac{2φy\_{1,3}}{ρ}+2\left(\hat{δ}\_{1}^{2}+1\right)\left(2\hat{δ}\_{1}y\_{1,3}-y\_{1,3}^{2}\right)-\left(2\hat{δ}\_{1}y\_{1,3}-y\_{1,3}^{2}\right)^{2}<0$

(A183)

We then substitute $ A$ for $\hat{y}\_{1,3} $to obtain:

$-A\hat{y}\_{1,2}\left(\hat{δ}\_{1}^{2}+1\right)-\frac{2φA}{ρ}+2A\left(\hat{δ}\_{1}^{2}+1\right)\left(2\hat{δ}\_{1}-A\right)-A^{2}\left(2\hat{δ}\_{1}-A\right)^{2}<0$ (A184)

Which can be simplified to:

$A\left(\hat{y}\_{1,2}\left(\hat{δ}\_{1}^{2}+1\right)-A\left(2\hat{δ}\_{1}-A\right)^{2}\right)<0$ (A185)

Since

$A=Q+\frac{\hat{y}\_{1,2}}{2}-\frac{\sqrt{\hat{Q}\left(\frac{φ}{2ρ}-\hat{Q}\right)}}{Q}$ (A186)

And

$2\hat{δ}\_{1}-y\_{1,3}=Q+\frac{\hat{y}\_{1,2}}{2}+\frac{\sqrt{\hat{Q}\left(\frac{φ}{2ρ}-\hat{Q}\right)}}{Q}$ (A187)

We also have that

$\left(2\hat{δ}\_{1}-A\right)^{2}=\hat{δ}\_{1}^{2}+2\hat{δ}\_{1}\frac{\sqrt{\hat{Q}\left(\frac{φ}{2ρ}-\hat{Q}\right)}}{Q}+\frac{\hat{Q}\left(\frac{φ}{2ρ}-\hat{Q}\right)}{Q^{2}}$ (A188)

If $A>0$, then since $\hat{y}\_{1,2}<0$, it follows that

$\hat{y}\_{1,2}\left(\hat{δ}\_{1}^{2}+1\right)-A\left(2\hat{δ}\_{1}-A\right)^{2}<0$ (A189)

And

$A\left(\hat{y}\_{1,2}\left(\hat{δ}\_{1}^{2}+1\right)-A\left(2\hat{δ}\_{1}-A\right)^{2}\right)<0$ (A190)

The equilibrium therefore would be stable; however, since $\hat{y}\_{1,2}<0$ and $A>0$, then $\hat{δ}\_{1}<\hat{δ}\_{3}$. Therefore the assumption that $\hat{δ}\_{1}$ controls water levels would not hold, and this solution is invalid.

However, for $A<0$

$A\left(\hat{y}\_{1,2}\left(\hat{δ}\_{1}^{2}+1\right)-A\left(2\hat{δ}\_{1}-A\right)^{2}\right)<0$ (A191)

Is true if

$\hat{y}\_{1,2}\left(\hat{δ}\_{1}^{2}+1\right)-A\left(2\hat{δ}\_{1}-A\right)^{2}>0$ (A192)

Which can also be expressed as

$\hat{y}\_{1,2}>\frac{A\left(2\hat{δ}\_{1}-A\right)^{2}}{\left(\hat{δ}\_{1}^{2}+1\right)}$ (A193)

Given $\hat{y}\_{1,2}>A$, it follows that

$2\hat{Q}+\hat{y}\_{1,2}-A>2\hat{Q}$ (A194)

And therefore

$\left(2\hat{δ}\_{1}-A\right)^{2}>4\hat{Q}^{2}$ (A195)

Given $A<\hat{y}\_{1,2}$, it also follows that

$Q-\frac{\sqrt{\hat{Q}\left(\frac{φ}{2ρ}-\hat{Q}\right)}}{Q}<\frac{\hat{y}\_{1,2}}{2}$ (A196)

From which we can obtain

$2\hat{Q}^{2}>Q\hat{y}\_{1,2}+2\sqrt{\hat{Q}\left(\frac{φ}{2ρ}-\hat{Q}\right)}$ (A197)

And:

$4\hat{Q}^{2}>Q\hat{y}\_{1,2}+2\hat{Q}^{2}+2\sqrt{\hat{Q}\left(\frac{φ}{2ρ}-\hat{Q}\right)}$ (A198)

Since

$\hat{δ}\_{1}^{2}+1=Q\hat{y}\_{1,2}+2\hat{Q}^{2}+2\sqrt{\hat{Q}\left(\frac{φ}{2ρ}-\hat{Q}\right)}$ (A199)

Combining Eqs. A195, A198, and A199, we have that

$\left(2\hat{δ}\_{1}-A\right)^{2}>\hat{δ}\_{1}^{2}+1>0$ (A200)

Since

$0<-\hat{y}\_{1,2}<-A$ (A201)

it follows that

$-A\left(2\hat{δ}\_{1}-A\right)^{2}>-\hat{y}\_{1,2}\left(\hat{δ}\_{1}^{2}+1\right)>0$ (A202)

and

$\hat{y}\_{1,2}\left(\hat{δ}\_{1}^{2}+1\right)-A\left(2\hat{δ}\_{1}-A\right)^{2}>0$ (A203)

Therefore the equilibrium at $\hat{y}\_{1,3}=A$ is stable whenever $\hat{y}\_{1,2}>A$.

*Longitudinal coupling downstream of slough*

We now consider a fourth patch, with depth *D4* that is located downstream of patch 2.

$\frac{dD\_{4}}{dt}=\hat{y}\_{2,4}\left(φ\frac{2\hat{δ}\_{2}-y\_{2,4}}{\left(\hat{δ}\_{2}^{2}+1\right)\left(\left(\hat{δ}\_{2}-\hat{y}\_{2,4}\right)^{2}+1\right)}-ρ\right)$ (A203)

Which is entirely equivalent to Eq. A141. Therefore equilibria exist at:

$\hat{y}\_{2,4}=0$ (A204)

$∴\hat{δ}\_{4}=\hat{δ}\_{2}$ (A205)

And where

$φ\frac{2\hat{δ}\_{2}-y\_{2,4}}{\left(\hat{δ}\_{2}^{2}+1\right)\left(\left(\hat{δ}\_{2}-\hat{y}\_{2,4}\right)^{2}+1\right)}-ρ=0$ (A206)

The latter solutions require that $\hat{δ}\_{4}<\hat{δ}\_{2}$, in which the assumption that $\hat{δ}\_{1}$ and $\hat{δ}\_{2}$ control water levels would not hold, and this solution is invalid. The trivial solution is therefore the only stable solution (i.e., downstream of a slough only another slough is stable).