Heffernan, J. B., D. L. Watts, and M. J. Cohen. Discharge competence as an ecohydrologic mechanism for pattern formation in peatlands: a meta-ecosystem model of the Everglades ridge-slough landscape.

**Supporting Information: Model Analysis S1**

*Local C balance and elevation change:*

Water depth (D) is the difference between surface water level (h) and soil elevation (z):

(A1)

So that:

(A2)

Local changes in soil elevation are driven by the balance of primary productivity (P) and decomposition (R):

(A3)

We model primary production with water depth as a function of water depth. In this model, gross primary production (P) has a maxima at the long-term mean water depth that is optimum for sawgrass growth (σ), and declines with increasing and decreasing depth:

(A4)

where *P*s is the gross peat production in sloughs, *P*r is gross peat production at the optimal depth for sawgrass growth, and *DT* is the depth at which peat accretion is the average of these two end-members (analogous to a half-saturation constant in monod kinetics).

Respiration declines with increasing water depth:

(A5)

where *Rσ* is the rate of gross peat decomposition when water levels are at the optimum depth for sawgrass growth, and rd is the rate of respiration decline with depth.

We now define as water depth indexed to have a value of 0 when :

(A6)

Substituting Eq. A6 in to expressions for productivity and respiration yields:

(A7)

and

(A8)

Changes in elevation can therefore be described by:

(A9)

Let

(A10)

And

(A11)

Substituting Eq. A11 into Eq. A9 yields

(A12)

Which simplifies to:

(A13)

We now let

(A14)

(A15)

So that Eq. A13 can be simplified to:

(A16)

Substituting and simplifying yields

(A17)

When , the change in elevation is

(A18)

So that

(A19)

*Lateral coupling*

Local changes in depth resulting from peat accretion and changes in water level are described by:

(A20)

In our lateral coupling model, water must be routed through the shared cross section of the two adjacent patches, which therefore control water level:

(A21)

Substituting in to Eq. A2 gives changes in depth for each patch in terms of changes in elevation of both patches:

(A22)

(A23)

Any changes in depth in the two patches are thus equal in magnitude and opposite in sign. Combining Eqs. A20 and A23, we obtain:

(A24)

Depth equilibria in the two patches therefore occur when:

(A25)

Which is true when:

(A26)

The depth in either patch is determined by discharge (*q*), which is a boundary condition, and velocity (*v*), which is assumed to be constant:

(A27)

The sum of discharges in the two patches is therefore:

(A28)

When water depths in each patch are indexed to optimum depth for productivity (*σ*), Eq. A28 is equivalent to:

(A29)

We now define as the difference in elevation between the two laterally-adjacent patches, so that:

(A30)

(A31)

(A32)

Combining Eq. A6 and A29, we have:

(A33)

We now define

(A34)

So that:

(A35)

From which it follows that:

(A36)

Let be discharge, indexed to sawgrass optimum depth and scaled to

(A37)

It then follows from Eqs. A36 and A37 that:

(A38)

We now define as the difference in elevation so that:

(A39)

(A40)

(A41)

Substituting these identities into Eq. A26 yields:

(A42)

Combining Eqs. A38 and A41 yields:

(A43)

From which we have that:

(A44)

(A45)

We now substitute Eqs. A44 and A45 into Eq. A42 to obtain the final form of the equation describing changes in depth:

(A46)

*Equilibrium solutions for lateral coupling model*

Depth equilibria occur when

(A47)

Which is equivalent to:

(A48)

Eq. A48 is true when:

(A49)

Or

(A50)

Eq. A50 may be expressed as a quadratic in :

(A51)

Which has solutions:

(A52)

where

(A53)

(A54)

(A55)

Since

(A56)

and

(A57)

It follows that:

(A58)

Which can simplified to:

(A59)

(A60)

Therefore solutions to Eq. A48 are:

(A61)

Since the designation of patches 1 and 2 are entirely arbitrary, we take the positive and negative roots for values of solutions to be equivalent solutions. Equilibrium depths for laterally-adjacent patches therefore occur when:

(A62)

*Bounding existence of solutions to lateral coupling model*

The trivial solution to Eq. A46 at always exists.

Solutions at

(A63)

Exist only if

(A64)

Which requires

(A65)

Assuming , there is always some Q for which this is true.

The solution at

(A66)

Further requires that

(A67)

(A68)

This is true for , over which range a real solution to Eq. A66 always exists.

For , a real solution to Eq. A66 exists when:

(A69)

Which is equivalent to:

(A70)

Since the value of Eq. A70 at the limit is positive, a real solution only exists for if

(A71)

Has real solutions. The discriminant for a quartic function (is:

(A72)

Eq. A71 has real solutions if . For Eq. A71, parameters of the discriminant are:

(A73)

(A74)

(A75)

(A76)

(A77)

Substituting these parameters in to the generic determinant formula yields:

(A78)

Which simplifies to:

(A79)

(A80)

Eq. A71 has real solutions when:

(A81)

Which is true only when:

(A82)

The actual value of the solutions to Eq. A70, which determine the discharge at which bifurcations occur, are given by Eqs. A110-112. We present analyses of these solutions after some preliminary analysis of the other nonzero solution.

The solution at

(A83)

Is real if

(A84)

Which is never true for . Since , it follows that a real solution exists when .

Assuming , Eq. A83 has a real solution when

(A85)

Which is equivalent to:

(A86)

Given the generic quartic equation:

(A87)

its solution can be found by means of the following calculations (Cardano’s solution). Let:

(A88)

(A89)

(A90)

For Eq. A86,

(A91)

(A92)

(A93)

We now let:

(A94)

(A95)

(A96)

For Eq. A86,

(A97)

(A98)

(A99)

We now let

(A100)

And let

(A101)

So that

(A102)

We further let

(A103)

So that

(A104)

Solutions to the quartic equation are given by:

(A105)

For Eq. A86, the root

(A106)

Is equivalent to

(A107)

(A108)

(A109)

Solutions to Eq. A86 occur at

(A110)

Where

(A111)

And

(A112)

**Summary of bifurcations**

The upper bound of both equilibria is .

The lower bound of the equilibrium depth difference

(A113)

Requires , and so occurs when

(A114)

This solution exists only if

The lower bound of the equilibrium depth difference

(A115)

Requires , and so occurs when

(A116)

This solution exists only when

*Stability of solutions to lateral coupling model*

To assess the stability of equilibrium depth differences, we take the derivative of Eq. A46 with respect to , which is:

(A117)

Evaluating this derivative for yields:

(A118)

The equilibrium is stable when

(A119)

Which is equivalent to

(A120)

So the equilibrium at is unstable between the lower bounds (bifurcations) for the other two equilibria.

The derivative of Eq. A46 can also be expressed as:

(A121)

Eq. A51 can be re-arranged as:

(A122)

Which we substitute in to Eq. A121:

(A123)

This expression simplifies to:

(A124)

Evaluating for the equilibrium given by Eq. A62, we have that

(A125)

Which we substitute into Eq. A124 to obtain:

(A126)

Since both the numerator and the denominator are always positive, the equilibrium at

(A127)

Is always stable when it exists.

Evaluating for the equilibrium given by Eq. A62, we have that

(A128)

Which we substitute into Eq. A124 to obtain:

(A129)

Since the numerator and denominator are always positive, the equilibrium at

(A130)

Is always unstable when it exists.

*Longitudinal coupling*

We now consider a third patch, with soil elevaton , located downstream of the deeper of two patches adjacent upstream patches (i.e. and . Further, we assume that and at equilibrium, meaning that

(A131)

or

(A132)

Since the equilibrium at

(A133)

Is unstable and so unlikely to persist as a constraint on water levels to patch . We take as given that is the higher elevation (shallower depth) of the two upstream patches.

Changes in depth are governed by the same carbon balance responses to water depth as the upstream patches

(A134)

However, we assume that so that does not control but only responds to water levels.

Changes in water level for the downstream patch () are dependent on the changes in elevation (and thus water level) of the two upstream patches:

(A135)

So that changes in depth for the downstream patch are governed by:

(A136)

Letting

(A137)

(A138)

(A139)

(A140)

Eq. A136 can be simplified to:

(A141)

Equilibria occur where

(A142)

The trivial solution ( is equivalent to:

(A143)

Additional solutions occur when:

(A144)

Since follows same rules as , it seems intuitive that an equilibrium would also occur when

(A145)

From which it follows that

(A146)

and

(A147)

So that Eq. A141 would be equivalent to:

(A148)

For which

(A149)

has already been shown to be a solution (Eqs. A47-A62), so this is also a solution for .

(A150)

Can be expressed as:

(A151)

Which is a quadratic in . So there remains one additional root, which we will obtain by factoring out the preceding solution. First, we selectively substitute for:

(A152)

Which leaves us

(A153)

Let A be such that:

(A154)

(A155)

(A156)

Therefore

(A157)

Is the third solution to Eq. A141.

*Assessing stability of longitudinally-coupled patches*

To assess the stability of equilibrium depth differences, we take the derivative of Eq. A141, which is:

(A158)

Equilibria are stable when:

(A159)

When

(A160)

Given

(A161)

It follows that

(A162)

From which we have that

(A163)

And is therefore negative when . The equilibrium at is stable whenever , and therefore whenever it exists.

To assess stability of other equilibria, we first expand Eq. A141:

(A164)

And substitute using Eq. A156

(A165)

When ,

(A166)

Substituting based on Eq. A156, we have that:

(A167)

Which is true when:

(A168)

To evaluate this identity, we begin with

(A169)

At equilibrium for

(A170)

Which can also be expressed as

(A171)

Eq. A156 is therefore equivalent to:

(A172)

Since

(A173)

Eq. A156 is also equivalent to:

(A174)

and

(A175)

If we assume

(A176)

Then it follows that

(A177)

(A178)

(A179)

and therefore that

(A180)

Since

(A181)

Will be true whenever . The equilibrium where will therefore be stable when

, in which case it is the deepest of the three equilibria.

To assess stability of the third equilibrium (, we again begin by expanding Eq. A141 to:

(A182)

And substitute in based on Eq. A156.

(A183)

We then substitute for to obtain:

(A184)

Which can be simplified to:

(A185)

Since

(A186)

And

(A187)

We also have that

(A188)

If , then since , it follows that

(A189)

And

(A190)

The equilibrium therefore would be stable; however, since and , then . Therefore the assumption that controls water levels would not hold, and this solution is invalid.

However, for

(A191)

Is true if

(A192)

Which can also be expressed as

(A193)

Given , it follows that

(A194)

And therefore

(A195)

Given , it also follows that

(A196)

From which we can obtain

(A197)

And:

(A198)

Since

(A199)

Combining Eqs. A195, A198, and A199, we have that

(A200)

Since

(A201)

it follows that

(A202)

and

(A203)

Therefore the equilibrium at is stable whenever .

*Longitudinal coupling downstream of slough*

We now consider a fourth patch, with depth *D4* that is located downstream of patch 2.

(A203)

Which is entirely equivalent to Eq. A141. Therefore equilibria exist at:

(A204)

(A205)

And where

(A206)

The latter solutions require that , in which the assumption that and control water levels would not hold, and this solution is invalid. The trivial solution is therefore the only stable solution (i.e., downstream of a slough only another slough is stable).