

S3 Appendix Additional Proofs

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1 The one-way FE estimator is a weighted average of coefficient estimates within each case/time point

Proposition 1. Let β_i be the coefficient on x_{it} from the regression $y_{it} = \alpha_i + \beta_i x_{it} + \varepsilon_{it}$ that only uses the observations for which the case is i . Without loss of generality, suppose case i is observed for T_i consecutive time points, where the time index t is labeled such that $t \in \{1, 2, \dots, T_i\}$. By OLS, the coefficient is estimated by

$$\hat{\beta}_i = \frac{\sum_{t=1}^{T_i} (x_{it} - \bar{x}_i)(y_{it} - \bar{y}_i)}{\sum_{t=1}^{T_i} (x_{it} - \bar{x}_i)^2}. \quad (1)$$

The one-way FE estimator is a weighted average of coefficient estimates within each case:

$$\hat{\beta}_{\text{caseFE}} = \frac{\sum_{i=1}^N \omega_i \hat{\beta}_i}{\sum_{i=1}^N \omega_i}, \quad (2)$$

where

$$\omega_i = \sum_{t=1}^{T_i} (x_{it} - \bar{x}_i)^2 = T_i \times V(x_{it}). \quad (3)$$

Proof. The one-way FE estimator is

$$\hat{\beta}_{\text{caseFE}} = \frac{\sum_{i=1}^N \sum_{t=1}^{T_i} (x_{it} - \bar{x}_i)(y_{it} - \bar{y}_i)}{\sum_{i=1}^N \sum_{t=1}^{T_i} (x_{it} - \bar{x}_i)^2}. \quad (4)$$

We can multiply and divide a factor of $\sum_{t=1}^{T_i} (x_{it} - \bar{x}_i)^2$ within the summation across cases in the numerator,

$$\hat{\beta}_{\text{caseFE}} = \frac{\sum_{i=1}^N \sum_{t=1}^{T_i} (x_{it} - \bar{x}_i)^2 \frac{\sum_{t=1}^{T_i} (x_{it} - \bar{x}_i)(y_{it} - \bar{y}_i)}{\sum_{t=1}^{T_i} (x_{it} - \bar{x}_i)^2}}{\sum_{i=1}^N \sum_{t=1}^{T_i} (x_{it} - \bar{x}_i)^2}, \quad (5)$$

which can be rewritten as

$$\hat{\beta}_{\text{caseFE}} = \frac{\sum_{i=1}^N \omega_i \hat{\beta}_i}{\sum_{i=1}^N \omega_i} \quad (6)$$

where

$$\omega_i = \sum_{t=1}^{T_i} (x_{it} - \bar{x}_i)^2 = T_i \times V(x_{it}). \quad \blacksquare \quad (7)$$

Similarly, the one-way time FE estimator is equal to a weighted average of coefficients derived from OLS on the data subsetting by time point. This proof follows the preceding proof exactly.

2 Derivation of the Two-Way Fixed Effects Estimator for Balanced Panels

Our objective in this section is to derive a clear formula for the two-way FE estimator in balanced panels. A formula, like the well known formula for case FEs in equation 3 in the main text, sheds light on the behavior of this estimator with regard to how it uses the cross-sectional and time variation in the data. Note that this formula builds on the two-way FE transformation in equation 8 in the main text, which is only valid for balanced panels. Therefore this formula is also only valid in the case of balanced panels.

We begin by proving a number of lemmas that we use in the proof of proposition 2 below.

Lemma 1a. The following equivalence holds in both balanced and unbalanced panels:

$$\sum_{i=1}^N \sum_{t=1}^{T_i} (x_{it} - \bar{x}_i)(y_{it} - \bar{y}_i) = \sum_{i=1}^N \sum_{t=1}^{T_i} (x_{it} - \bar{x}_i)(y_{it} - \bar{y}) = \sum_{i=1}^N \sum_{t=1}^{T_i} (x_{it} - \bar{x})(y_{it} - \bar{y}_i). \quad (8)$$

Proof. We demonstrate that all three expressions are individually equal to $\sum_{i=1}^N \sum_{t=1}^{T_i} x_{it}y_{it} - \sum_{i=1}^N T_i \bar{x}_i \bar{y}_i$ and are therefore equal to each other:

$$\begin{aligned} \sum_{i=1}^N \sum_{t=1}^{T_i} (x_{it} - \bar{x}_i)(y_{it} - \bar{y}_i) &= \sum_{i=1}^N \sum_{t=1}^{T_i} (x_{it}y_{it} - x_{it}\bar{y}_i - \bar{x}_iy_{it} + \bar{x}_i\bar{y}_i) \\ &= \sum_{i=1}^N \sum_{t=1}^{T_i} x_{it}y_{it} - \sum_{i=1}^N \sum_{t=1}^{T_i} x_{it}\bar{y}_i - \sum_{i=1}^N \sum_{t=1}^{T_i} \bar{x}_iy_{it} + \sum_{i=1}^N \sum_{t=1}^{T_i} \bar{x}_i\bar{y}_i \\ &= \sum_{i=1}^N \sum_{t=1}^{T_i} x_{it}y_{it} - \sum_{i=1}^N \bar{y}_i \sum_{t=1}^{T_i} x_{it} - \sum_{i=1}^N \bar{x}_i \sum_{t=1}^{T_i} y_{it} + \sum_{i=1}^N T_i \bar{x}_i \bar{y}_i \\ &= \sum_{i=1}^N \sum_{t=1}^{T_i} x_{it}y_{it} - \sum_{i=1}^N \bar{y}_i (T_i \bar{x}_i) - \sum_{i=1}^N \bar{x}_i (T_i \bar{y}_i) + \sum_{i=1}^N T_i \bar{x}_i \bar{y}_i \\ &= \sum_{i=1}^N \sum_{t=1}^{T_i} x_{it}y_{it} - \sum_{i=1}^N T_i \bar{x}_i \bar{y}_i. \end{aligned} \quad (9)$$

$$\begin{aligned}
\sum_{i=1}^N \sum_{t=1}^{T_i} (x_{it} - \bar{x}_i)(y_{it} - \bar{y}) &= \sum_{i=1}^N \sum_{t=1}^{T_i} (x_{it}y_{it} - x_{it}\bar{y} - \bar{x}_iy_{it} + \bar{x}_i\bar{y}) \\
&= \sum_{i=1}^N \sum_{t=1}^{T_i} x_{it}y_{it} - \sum_{i=1}^N \sum_{t=1}^{T_i} x_{it}\bar{y} - \sum_{i=1}^N \sum_{t=1}^{T_i} \bar{x}_iy_{it} + \sum_{i=1}^N \sum_{t=1}^{T_i} \bar{x}_i\bar{y} \\
&= \sum_{i=1}^N \sum_{t=1}^{T_i} x_{it}y_{it} - \bar{y} \sum_{i=1}^N \sum_{t=1}^{T_i} x_{it} - \sum_{i=1}^N \bar{x}_i \sum_{t=1}^{T_i} y_{it} + \bar{y} \sum_{i=1}^N T_i \bar{x}_i \\
&= \sum_{i=1}^N \sum_{t=1}^{T_i} x_{it}y_{it} - \bar{y} \sum_{i=1}^N T_i \bar{x}_i - \sum_{i=1}^N \bar{x}_i (T_i \bar{y}_i) + \bar{y} \sum_{i=1}^N T_i \bar{x}_i \\
&= \sum_{i=1}^N \sum_{t=1}^{T_i} x_{it}y_{it} - \sum_{i=1}^N T_i \bar{x}_i \bar{y}_i.
\end{aligned} \tag{10}$$

$$\begin{aligned}
\sum_{i=1}^N \sum_{t=1}^{T_i} (x_{it} - \bar{x})(y_{it} - \bar{y}_i) &= \sum_{i=1}^N \sum_{t=1}^{T_i} (x_{it}y_{it} - x_{it}\bar{y}_i - \bar{x}y_{it} + \bar{x}\bar{y}_i) \\
&= \sum_{i=1}^N \sum_{t=1}^{T_i} x_{it}y_{it} - \sum_{i=1}^N \sum_{t=1}^{T_i} x_{it}\bar{y}_i - \sum_{i=1}^N \sum_{t=1}^{T_i} \bar{x}y_{it} + \sum_{i=1}^N \sum_{t=1}^{T_i} \bar{x}\bar{y}_i \\
&= \sum_{i=1}^N \sum_{t=1}^{T_i} x_{it}y_{it} - \sum_{i=1}^N \bar{y}_i \sum_{t=1}^{T_i} x_{it} - \bar{x} \sum_{i=1}^N \sum_{t=1}^{T_i} y_{it} + \bar{x} \sum_{i=1}^N T_i \bar{y}_i \\
&= \sum_{i=1}^N \sum_{t=1}^{T_i} x_{it}y_{it} - \sum_{i=1}^N \bar{y}_i (T_i \bar{x}_i) - \bar{x} \sum_{i=1}^N T_i \bar{y}_i + \bar{x} \sum_{i=1}^N T_i \bar{y}_i \\
&= \sum_{i=1}^N \sum_{t=1}^{T_i} x_{it}y_{it} - \sum_{i=1}^N T_i \bar{x}_i \bar{y}_i. \quad \blacksquare
\end{aligned} \tag{11}$$

Lemma 1b. The following equivalence holds in both balanced and unbalanced panels:

$$\sum_{i=1}^N \sum_{t=1}^{T_i} (x_{it} - \bar{x}_t)(y_{it} - \bar{y}_t) = \sum_{i=1}^N \sum_{t=1}^{T_i} (x_{it} - \bar{x}_t)(y_{it} - \bar{y}) = \sum_{i=1}^N \sum_{t=1}^{T_i} (x_{it} - \bar{x})(y_{it} - \bar{y}_t). \tag{12}$$

Proof. The proof follows the proof of lemma 1a, substituting $\bar{x}_t$ for $\bar{x}_i$ and $\bar{y}_t$ for $\bar{y}_i$ and rewriting the summation as $\sum_{t=1}^T \sum_{i=1}^{N_t}$. $\blacksquare$

Lemma 2a. The following equivalence holds in both balanced and unbalanced panels:

$$\sum_{i=1}^N \sum_{t=1}^{T_i} (x_{it} - \bar{x}_i)^2 = \sum_{i=1}^N \sum_{t=1}^{T_i} (x_{it} - \bar{x}_i)(x_{it} - \bar{x}). \tag{13}$$

Proof. We demonstrate that both expressions are individually equal to $\sum_{i=1}^N \sum_{t=1}^{T_i} x_{it}^2 - \sum_{i=1}^N T_i \bar{x}_i^2$ and

are therefore equal to each other:

$$\begin{aligned}
\sum_{i=1}^N \sum_{t=1}^{T_i} (x_{it} - \bar{x}_i)^2 &= \sum_{i=1}^N \sum_{t=1}^{T_i} (x_{it}^2 - 2x_{it}\bar{x}_i + \bar{x}_i^2) \\
&= \sum_{i=1}^N \sum_{t=1}^{T_i} x_{it}^2 - 2 \sum_{i=1}^N \sum_{t=1}^{T_i} x_{it}\bar{x}_i + \sum_{i=1}^N \sum_{t=1}^{T_i} \bar{x}_i^2 \\
&= \sum_{i=1}^N \sum_{t=1}^{T_i} x_{it}^2 - 2 \sum_{i=1}^N \bar{x}_i \sum_{t=1}^{T_i} x_{it} + \sum_{i=1}^N T_i \bar{x}_i^2 \\
&= \sum_{i=1}^N \sum_{t=1}^{T_i} x_{it}^2 - 2 \sum_{i=1}^N T_i \bar{x}_i^2 + \sum_{i=1}^N T_i \bar{x}_i^2 \\
&= \sum_{i=1}^N \sum_{t=1}^{T_i} x_{it}^2 - \sum_{i=1}^N T_i \bar{x}_i^2.
\end{aligned} \tag{14}$$

$$\begin{aligned}
\sum_{i=1}^N \sum_{t=1}^{T_i} (x_{it} - \bar{x}_i)(x_{it} - \bar{x}) &= \sum_{i=1}^N \sum_{t=1}^{T_i} (x_{it}^2 - \bar{x}x_{it} - \bar{x}_i x_{it} + \bar{x}_i \bar{x}) \\
&= \sum_{i=1}^N \sum_{t=1}^{T_i} x_{it}^2 - \sum_{i=1}^N \sum_{t=1}^{T_i} \bar{x}x_{it} - \sum_{i=1}^N \sum_{t=1}^{T_i} \bar{x}_i x_{it} + \sum_{i=1}^N \sum_{t=1}^{T_i} \bar{x}_i \bar{x} \\
&= \sum_{i=1}^N \sum_{t=1}^{T_i} x_{it}^2 - \bar{x} \sum_{i=1}^N \sum_{t=1}^{T_i} x_{it} - \sum_{i=1}^N \bar{x}_i \sum_{t=1}^{T_i} x_{it} + \bar{x} \sum_{i=1}^N T_i \bar{x}_i \\
&= \sum_{i=1}^N \sum_{t=1}^{T_i} x_{it}^2 - \bar{x} \sum_{i=1}^N T_i \bar{x}_i - \sum_{i=1}^N T_i \bar{x}_i^2 + \bar{x} \sum_{i=1}^N T_i \bar{x}_i \\
&= \sum_{i=1}^N \sum_{t=1}^{T_i} x_{it}^2 - \sum_{i=1}^N T_i \bar{x}_i^2. \quad \blacksquare
\end{aligned} \tag{15}$$

Lemma 2b. The following equivalence holds in both balanced and unbalanced panels:

$$\sum_{i=1}^N \sum_{t=1}^{T_i} (x_{it} - \bar{x}_t)^2 = \sum_{i=1}^N \sum_{t=1}^{T_i} (x_{it} - \bar{x}_t)(x_{it} - \bar{x}). \tag{16}$$

Proof. The proof follows the proof of lemma 2a, substituting $\bar{x}_t$ for $\bar{x}_i$ and rewriting the summation as $\sum_{t=1}^T \sum_{i=1}^{N_t}$. $\blacksquare$

Lemma 3. If the panels in the data are balanced, then

$$\sum_{i=1}^N \sum_{t=1}^T (x_{it} - \bar{x}_i)(y_{it} - \bar{y}_t) = \sum_{i=1}^N \sum_{t=1}^T (x_{it} - \bar{x}_t)(y_{it} - \bar{y}_i). \tag{17}$$

Proof. This proof depends on the fact that

$$\bar{x} = \frac{\sum_{i=1}^N \sum_{t=1}^T x_{it}}{NT} = \frac{\sum_{i=1}^N \bar{x}_i}{N} = \frac{\sum_{t=1}^T \bar{x}_t}{T} \tag{18}$$

if and only if the panels are balanced:

$$\begin{aligned}
\sum_{i=1}^N \sum_{t=1}^T (x_{it} - \bar{x}_i)(y_{it} - \bar{y}_t) &= \sum_{i=1}^N \sum_{t=1}^T (x_{it}y_{it} - x_{it}\bar{y}_t - \bar{x}_iy_{it} + \bar{x}_i\bar{y}_t) \\
&= \sum_{i=1}^N \sum_{t=1}^T x_{it}y_{it} - \sum_{i=1}^N \sum_{t=1}^T x_{it}\bar{y}_t - \sum_{i=1}^N \sum_{t=1}^T \bar{x}_iy_{it} + \sum_{i=1}^N \sum_{t=1}^T \bar{x}_i\bar{y}_t \\
&= \sum_{i=1}^N \sum_{t=1}^T x_{it}y_{it} - \sum_{t=1}^T \bar{y}_t \sum_{i=1}^N x_{it} - \sum_{i=1}^N \bar{x}_i \sum_{t=1}^T y_{it} + \sum_{i=1}^N \bar{x}_i \sum_{t=1}^T \bar{y}_t \\
&= \sum_{i=1}^N \sum_{t=1}^T x_{it}y_{it} - N \sum_{t=1}^T \bar{x}_t \bar{y}_t - T \sum_{i=1}^N \bar{x}_i \bar{y}_i + \sum_{i=1}^N \bar{x}_i \sum_{t=1}^T \bar{y}_t \\
&= \sum_{i=1}^N \sum_{t=1}^T x_{it}y_{it} - N \sum_{t=1}^T \bar{x}_t \bar{y}_t - T \sum_{i=1}^N \bar{x}_i \bar{y}_i + NT \bar{x} \bar{y} \\
&= \sum_{i=1}^N \sum_{t=1}^T x_{it}y_{it} - \sum_{t=1}^T \bar{x}_t \sum_{i=1}^N y_{it} - \sum_{i=1}^N \sum_{t=1}^T x_{it} \bar{y}_i + \sum_{i=1}^N \bar{x}_i \sum_{t=1}^T \bar{y}_i \\
&= \sum_{i=1}^N \sum_{t=1}^T (x_{it}y_{it} - \bar{x}_t y_{it} - x_{it} \bar{y}_i + \bar{x}_t \bar{y}_i) \\
&= \sum_{i=1}^N \sum_{t=1}^T (x_{it} - \bar{x}_t)(y_{it} - \bar{y}_i). \quad \blacksquare
\end{aligned} \tag{19}$$

Lemma 4. If the panels in the data are balanced, then

$$\sum_{i=1}^N \sum_{t=1}^T \left[(x_{it} - \bar{x}_i)(y_{it} - \bar{y}_i) + (x_{it} - \bar{x}_t)(y_{it} - \bar{y}_t) - (x_{it} - \bar{x}_i)(y_{it} - \bar{y}_t) \right] = \sum_{i=1}^N \sum_{t=1}^T (x_{it} - \bar{x})(y_{it} - \bar{y}). \tag{20}$$

Proof.

$$\begin{aligned}
& \sum_{i=1}^N \sum_{t=1}^T \left[(x_{it} - \bar{x}_i)(y_{it} - \bar{y}_i) + (x_{it} - \bar{x}_t)(y_{it} - \bar{y}_t) - (x_{it} - \bar{x}_i)(y_{it} - \bar{y}_t) \right] \\
&= \sum_{i=1}^N \sum_{t=1}^T \left[(x_{it}y_{it} - x_{it}\bar{y}_i - \bar{x}_iy_{it} + \bar{x}_i\bar{y}_i) + (x_{it}y_{it} - x_{it}\bar{y}_t - \bar{x}_ty_{it} + \bar{x}_t\bar{y}_t) - (x_{it}y_{it} - x_{it}\bar{y}_t - \bar{x}_iy_{it} + \bar{x}_i\bar{y}_t) \right] \\
&= \sum_{i=1}^N \sum_{t=1}^T \left[(x_{it}y_{it} + x_{it}y_{it} - x_{it}y_{it}) - x_{it}(\bar{y}_i + \bar{y}_t - \bar{y}_t) - (\bar{x}_i + \bar{x}_t - \bar{x}_i)y_{it} + \bar{x}_i\bar{y}_i + \bar{x}_t\bar{y}_t - \bar{x}_i\bar{y}_t \right] \\
&= \sum_{i=1}^N \sum_{t=1}^T \left[x_{it}y_{it} - x_{it}\bar{y}_i - \bar{x}_ty_{it} + \bar{x}_i\bar{y}_i + \bar{x}_t\bar{y}_t - \bar{x}_i\bar{y}_t \right] \\
&= \sum_{i=1}^N \sum_{t=1}^T x_{it}y_{it} - \sum_{i=1}^N \sum_{t=1}^T x_{it}\bar{y}_i - \sum_{i=1}^N \sum_{t=1}^T \bar{x}_ty_{it} + \sum_{i=1}^N \sum_{t=1}^T \bar{x}_i\bar{y}_i + \sum_{i=1}^N \sum_{t=1}^T \bar{x}_t\bar{y}_t - \sum_{i=1}^N \sum_{t=1}^T \bar{x}_i\bar{y}_t \\
&= \sum_{i=1}^N \sum_{t=1}^T x_{it}y_{it} - T \sum_{i=1}^N \bar{x}_i\bar{y}_i - N \sum_{t=1}^T \bar{x}_t\bar{y}_t + T \sum_{i=1}^N \bar{x}_i\bar{y}_i + N \sum_{t=1}^T \bar{x}_t\bar{y}_t - NT\bar{x}\bar{y} \\
&= \sum_{i=1}^N \sum_{t=1}^T x_{it}y_{it} - NT\bar{x}\bar{y} \\
&= \sum_{i=1}^N \sum_{t=1}^T x_{it}y_{it} - NT\bar{x}\bar{y} - NT\bar{x}\bar{y} + NT\bar{x}\bar{y} \\
&= \sum_{i=1}^N \sum_{t=1}^T x_{it}y_{it} - \sum_{i=1}^N \sum_{t=1}^T x_{it}\bar{y} - \sum_{i=1}^N \sum_{t=1}^T \bar{x}y_{it} + \sum_{i=1}^N \sum_{t=1}^T \bar{x}\bar{y} \\
&= \sum_{i=1}^N \sum_{t=1}^T (x_{it}y_{it} - x_{it}\bar{y} - \bar{x}y_{it} + \bar{x}\bar{y}) \\
&= \sum_{i=1}^N \sum_{t=1}^T (x_{it} - \bar{x})(y_{it} - \bar{y}). \quad \blacksquare
\end{aligned} \tag{21}$$

Lemma 5. If the panels in the data are balanced, then

$$\sum_{i=1}^N \sum_{t=1}^T \left[(x_{it} - \bar{x}_i)^2 + (x_{it} - \bar{x}_t)^2 - (x_{it} - \bar{x}_i)(x_{it} - \bar{x}_t) \right] = \sum_{i=1}^N \sum_{t=1}^T (x_{it} - \bar{x})^2. \tag{22}$$

Proof.

$$\begin{aligned}
& \sum_{i=1}^N \sum_{t=1}^T \left[(x_{it} - \bar{x}_i)^2 + (x_{it} - \bar{x}_t)^2 - (x_{it} - \bar{x}_i)(x_{it} - \bar{x}_t) \right] \\
&= \sum_{i=1}^N \sum_{t=1}^T \left[(x_{it}^2 - 2x_{it}\bar{x}_i + \bar{x}_i^2) + (x_{it}^2 - 2x_{it}\bar{x}_t + \bar{x}_t^2) - (x_{it}^2 - x_{it}\bar{x}_t - x_{it}\bar{x}_i + \bar{x}_i\bar{x}_t) \right] \\
&= \sum_{i=1}^N \sum_{t=1}^T \left[(x_{it}^2 + x_{it}^2 - x_{it}^2) - x_{it}(2\bar{x}_i + 2\bar{x}_t - \bar{x}_i - \bar{x}_t) + \bar{x}_i^2 + \bar{x}_t^2 - \bar{x}_i\bar{x}_t \right] \\
&= \sum_{i=1}^N \sum_{t=1}^T \left[x_{it}^2 - x_{it}\bar{x}_i + x_{it}\bar{x}_t + \bar{x}_i^2 + \bar{x}_t^2 - \bar{x}_i\bar{x}_t \right] \\
&= \sum_{i=1}^N \sum_{t=1}^T x_{it}^2 - \sum_{i=1}^N \sum_{t=1}^T x_{it}\bar{x}_i + \sum_{i=1}^N \sum_{t=1}^T x_{it}\bar{x}_t + \sum_{i=1}^N \sum_{t=1}^T \bar{x}_i^2 + \sum_{i=1}^N \sum_{t=1}^T \bar{x}_t^2 - \sum_{i=1}^N \sum_{t=1}^T \bar{x}_i\bar{x}_t \\
&= \sum_{i=1}^N \sum_{t=1}^T x_{it}^2 - T \sum_{i=1}^N \bar{x}_i^2 + N \sum_{t=1}^T \bar{x}_t^2 + T \sum_{i=1}^N \bar{x}_i^2 + N \sum_{t=1}^T \bar{x}_t^2 - NT\bar{x}^2 \\
&= \sum_{i=1}^N \sum_{t=1}^T x_{it}^2 - NT\bar{x}^2 \\
&= \sum_{i=1}^N \sum_{t=1}^T x_{it}^2 - NT\bar{x}^2 + NT\bar{x}^2 - NT\bar{x}^2 \\
&= \sum_{i=1}^N \sum_{t=1}^T x_{it}^2 - 2 \sum_{i=1}^N \sum_{t=1}^T x_{it}\bar{x} + \sum_{i=1}^N \sum_{t=1}^T \bar{x}^2 \\
&= \sum_{i=1}^N \sum_{t=1}^T (x_{it}^2 - 2x_{it}\bar{x} + \bar{x}^2) \\
&= \sum_{i=1}^N \sum_{t=1}^T (x_{it} - \bar{x})^2. \quad \blacksquare
\end{aligned} \tag{23}$$

Proposition 2. If the panels in the data are balanced, then the two-way FE estimator is given by

$$\hat{\beta}_{TW} = \frac{\sum_{i=1}^N \sum_{t=1}^T (x_{it} - \bar{x}_i)(y_{it} - \bar{y}_t)}{\sum_{i=1}^N \sum_{t=1}^T (x_{it} - \bar{x}_i)(x_{it} - \bar{x}_t)}, \tag{24}$$

or equivalently as

$$\hat{\beta}_{TW} = \frac{\sum_{i=1}^N \sum_{t=1}^T (x_{it} - \bar{x}_t)(y_{it} - \bar{y}_i)}{\sum_{i=1}^N \sum_{t=1}^T (x_{it} - \bar{x}_i)(x_{it} - \bar{x}_t)}. \tag{25}$$

Proof. The two-way fixed effects estimator is the OLS estimator applied to the following model,

$$y_{it}^* = \alpha_{TW} + \beta_{TW}x_{it}^* + \varepsilon_{it}, \tag{26}$$

where $y_{it}^* = y_{it} - \bar{y}_i - \bar{y}_t + \bar{y}$ and $x_{it}^* = x_{it} - \bar{x}_i - \bar{x}_t + \bar{x}$. By OLS, the coefficient in equation 26 is given by

$$\begin{aligned}
\hat{\beta}_{TW} &= \frac{\sum_{i=1}^N \sum_{t=1}^{T_i} (x_{it} - \bar{x}_i - \bar{x}_t + \bar{x}) (y_{it} - \bar{y}_i - \bar{y}_t + \bar{y})}{\sum_{i=1}^N \sum_{t=1}^{T_i} (x_{it} - \bar{x}_i - \bar{x}_t + \bar{x})^2} \\
&= \frac{\sum_{i=1}^N \sum_{t=1}^{T_i} (x_{it} + x_{it} - x_{it} - \bar{x}_i - \bar{x}_t + \bar{x}) (y_{it} + y_{it} - y_{it} - \bar{y}_i - \bar{y}_t + \bar{y})}{\sum_{i=1}^N \sum_{t=1}^{T_i} (x_{it} + x_{it} - x_{it} - \bar{x}_i - \bar{x}_t + \bar{x})^2} \\
&= \frac{\sum_{i=1}^N \sum_{t=1}^{T_i} [(x_{it} - \bar{x}_i) + (x_{it} - \bar{x}_t) - (x_{it} - \bar{x})] [(y_{it} - \bar{y}_i) + (y_{it} - \bar{y}_t) - (y_{it} - \bar{y})]}{\sum_{i=1}^N \sum_{t=1}^{T_i} [(x_{it} - \bar{x}_i) + (x_{it} - \bar{x}_t) - (x_{it} - \bar{x})]^2}, \\
&= \frac{\sum_{i=1}^N \sum_{t=1}^{T_i} A_{it}}{\sum_{i=1}^N \sum_{t=1}^{T_i} B_{it}}, \tag{27}
\end{aligned}$$

where

$$\begin{aligned}
A_{it} &= (x_{it} - \bar{x}_i)(y_{it} - \bar{y}_i) + (x_{it} - \bar{x}_i)(y_{it} - \bar{y}_t) - (x_{it} - \bar{x}_i)(y_{it} - \bar{y}) \\
&\quad + (x_{it} - \bar{x}_t)(y_{it} - \bar{y}_i) + (x_{it} - \bar{x}_t)(y_{it} - \bar{y}_t) - (x_{it} - \bar{x}_t)(y_{it} - \bar{y}) \\
&\quad - (x_{it} - \bar{x})(y_{it} - \bar{y}_i) - (x_{it} - \bar{x})(y_{it} - \bar{y}_t) + (x_{it} - \bar{x})(y_{it} - \bar{y}), \tag{28}
\end{aligned}$$

and

$$\begin{aligned}
B_{it} &= (x_{it} - \bar{x}_i)^2 + (x_{it} - \bar{x}_t)^2 + (x_{it} - \bar{x})^2 \\
&\quad + 2(x_{it} - \bar{x}_i)(x_{it} - \bar{x}_t) - 2(x_{it} - \bar{x}_i)(x_{it} - \bar{x}) - 2(x_{it} - \bar{x}_t)(x_{it} - \bar{x}). \tag{29}
\end{aligned}$$

From lemmas 1a, 1b, 2a, and 2b, A_{it} and B_{it} reduce to

$$\begin{aligned}
A_{it} &= (x_{it} - \bar{x})(y_{it} - \bar{y}) - (x_{it} - \bar{x}_i)(y_{it} - \bar{y}_i) - (x_{it} - \bar{x}_t)(y_{it} - \bar{y}_t) \\
&\quad + (x_{it} - \bar{x}_i)(y_{it} - \bar{y}_t) + (x_{it} - \bar{x}_t)(y_{it} - \bar{y}_i), \tag{30}
\end{aligned}$$

and

$$B_{it} = (x_{it} - \bar{x})^2 - (x_{it} - \bar{x}_i)^2 - (x_{it} - \bar{x}_t)^2 + 2(x_{it} - \bar{x}_i)(x_{it} - \bar{x}_t). \tag{31}$$

By lemma 3, A_{it} simplifies to

$$A_{it} = (x_{it} - \bar{x})(y_{it} - \bar{y}) - (x_{it} - \bar{x}_i)(y_{it} - \bar{y}_i) - (x_{it} - \bar{x}_t)(y_{it} - \bar{y}_t) + 2(x_{it} - \bar{x}_i)(y_{it} - \bar{y}_t), \tag{32}$$

or equivalently to

$$A_{it} = (x_{it} - \bar{x})(y_{it} - \bar{y}) - (x_{it} - \bar{x}_i)(y_{it} - \bar{y}_i) - (x_{it} - \bar{x}_t)(y_{it} - \bar{y}_t) + 2(x_{it} - \bar{x}_t)(y_{it} - \bar{y}_i). \tag{33}$$

By lemma 4, A_{it} further simplifies to

$$\begin{aligned}
A_{it} &= \left[(x_{it} - \bar{x}_i)(y_{it} - \bar{y}_i) + (x_{it} - \bar{x}_t)(y_{it} - \bar{y}_t) - (x_{it} - \bar{x}_i)(y_{it} - \bar{y}_t) \right] \\
&\quad - (x_{it} - \bar{x}_i)(y_{it} - \bar{y}_i) - (x_{it} - \bar{x}_t)(y_{it} - \bar{y}_t) + 2(x_{it} - \bar{x}_i)(y_{it} - \bar{y}_t) \\
&= (x_{it} - \bar{x}_i)(y_{it} - \bar{y}_t),
\end{aligned} \tag{34}$$

or equivalently to $(x_{it} - \bar{x}_t)(y_{it} - \bar{y}_i)$ by lemma 3. Applying lemma 5, B_{it} reduces as follows:

$$\begin{aligned}
B_{it} &= \left[(x_{it} - \bar{x}_i)^2 + (x_{it} - \bar{y}_t)^2 - (x_{it} - \bar{x}_i)(x_{it} - \bar{x}_t) \right] \\
&\quad - (x_{it} - \bar{x}_i)^2 - (x_{it} - \bar{x}_t)^2 + 2(x_{it} - \bar{x}_i)(x_{it} - \bar{x}_t) \\
&= (x_{it} - \bar{x}_i)(x_{it} - \bar{x}_t).
\end{aligned} \tag{35}$$

Therefore, the two-way fixed effect estimator is given by

$$\hat{\beta}_{TW} = \frac{\sum_{i=1}^N \sum_{t=1}^T (x_{it} - \bar{x}_i)(y_{it} - \bar{y}_t)}{\sum_{i=1}^N \sum_{t=1}^T (x_{it} - \bar{x}_i)(x_{it} - \bar{x}_t)}, \tag{36}$$

or equivalently as

$$\hat{\beta}_{TW} = \frac{\sum_{i=1}^N \sum_{t=1}^T (x_{it} - \bar{x}_t)(y_{it} - \bar{y}_i)}{\sum_{i=1}^N \sum_{t=1}^T (x_{it} - \bar{x}_i)(x_{it} - \bar{x}_t)}. \quad \blacksquare \tag{37}$$