

S2 Appendix

PDF of the length of a subvector of a unit vector.

Lemma 1. *Given a random variable X with probability distribution function $p_X(x)$, the probability distribution function of $Y = X^2$ is given by*

$$p_Y(y) = \frac{1}{2\sqrt{y}} \cdot p_X(\sqrt{y}). \quad (1)$$

Proof. Performing a change in variable with $y = f(x) = x^2$ gives

$$\begin{aligned} p_Y(y) &= \left| \frac{d}{dy} (f^{-1}(y)) \right| \cdot p_X(f^{-1}(y)) \\ &= \left| \frac{d}{dy} (\sqrt{y}) \right| \cdot p_X(\sqrt{y}) \\ &= \frac{1}{2\sqrt{y}} \cdot p_X(\sqrt{y}). \end{aligned} \quad (2)$$

□

Theorem 1. *Let $\mathbf{v} = (v_1, \dots, v_{n+m})$ be a vector of dimension $n+m$ with $n, m \in \mathbb{N}^+$ and random components distributed independently according to $v_i \sim \mathcal{N}(0; \sigma)$. Let $\hat{\mathbf{v}} = \mathbf{v}/|\mathbf{v}|$ be the vector normalized to unit length. Then the length $|\hat{\mathbf{v}}_{1:m}|$ of a subvector $\hat{\mathbf{v}}_{1:m} = (\hat{v}_1, \dots, \hat{v}_m)$ with m components is distributed according to the probability distribution function*

$$p_{SB}(x; n, m) = \frac{2}{B\left(\frac{n}{2}, \frac{m}{2}\right)} \left(x^2\right)^{(m-1)/2} \left(1-x^2\right)^{n/2-1}. \quad (3)$$

Proof. The length of the subvector is given by the random variable

$$\begin{aligned} |\hat{\mathbf{v}}_{1:m}| &= \frac{|\mathbf{v}_{1:m}|}{|\mathbf{v}|} \\ &= \frac{|\mathbf{v}_{1:m}|}{\sqrt{|\mathbf{v}_{1:m}|^2 + |\mathbf{v}_{m+1:n+m}|^2}} \\ &= \frac{1}{\sqrt{1 + \frac{|\mathbf{v}_{m+1:n+m}|^2}{|\mathbf{v}_{1:m}|^2}}} \\ &= \frac{1}{\sqrt{1 + Q}} \end{aligned} \quad (4)$$

with random variable

$$Q = \frac{|\mathbf{v}_{m+1:n+m}|^2}{|\mathbf{v}_{1:m}|^2}. \quad (5)$$

The probability distribution function of the quotient of two random variables is given by

$$p_{X/Y}(x) = \int_{-\infty}^{\infty} |y| p_X(xy) p_Y(y) dy \quad (6)$$

Using this equation, Theorem 1 in S1 Appendix, and Lemma 1 we obtain

$$\begin{aligned}
p_Q(x; \sigma, n, m) &= \int_{-\infty}^{\infty} |y| p_{|\mathbf{v}_{m+1:n+m}|^2}(xy; \sigma, n) p_{|\mathbf{v}_{1:m}|^2}(y; \sigma, m) dy \\
&= \int_0^{\infty} \frac{|y|}{4\sqrt{xy^2}} \cdot p_{|\mathbf{v}_{m+1:n+m}|^2}(\sqrt{xy}; \sigma, n) p_{|\mathbf{v}_{1:m}|}(\sqrt{y}; \sigma, m) dy \\
&= \int_0^{\infty} \frac{|y|}{4\sqrt{xy^2}} \cdot k_n k_m (\sqrt{xy})^{n-1} (\sqrt{y})^{m-1} \exp\left(-\frac{xy}{2\sigma^2}\right) \exp\left(-\frac{y}{2\sigma^2}\right) dy \\
&= \frac{k_n k_m}{4\sqrt{x}} \int_0^{\infty} x^{(n-1)/2} y^{(n+m-2)/2} \exp\left(-\frac{x+1}{2\sigma^2} \cdot y\right) dy \\
&= \frac{k_n k_m}{4} \cdot x^{n/2-1} \cdot \frac{2\sigma^2}{x+1} \int_0^{\infty} \underbrace{\frac{z+1}{2\sigma^2} y^{(n+m-2)/2}}_{\phi'(y)} \underbrace{\exp\left(-\frac{x+1}{2\sigma^2} \cdot y\right)}_{f(\phi(y))} dy.
\end{aligned} \tag{7}$$

Applying a substitution with $\phi(y) = \frac{x+1}{2\sigma^2} \cdot y$ and $f(\phi(y)) = \left(\frac{2\sigma^2}{x+1} \cdot \phi(y)\right)^{(n+m-2)/2} \cdot \exp(-\phi(y))$ gives

$$\begin{aligned}
p_Q(x; \sigma, n, m) &= \frac{k_n k_m}{4} \cdot x^{n/2-1} \cdot \frac{2\sigma^2}{x+1} \cdot \\
&\quad \int_{\phi(0)}^{\phi(\infty)} \left(\frac{2\sigma^2}{x+1} \cdot \phi(y)\right)^{(n+m-2)/2} \cdot \exp(-\phi(y)) dy \\
&= \frac{k_n k_m}{4} \cdot x^{n/2-1} \cdot \left(\frac{2\sigma^2}{x+1}\right)^{(n+m)/2} \int_0^{\infty} \phi(y)^{(n+m-2)/2} \cdot \exp(-\phi(y)) dy \\
&= \frac{k_n k_m}{4} \cdot x^{n/2-1} \cdot \left(\frac{2\sigma^2}{x+1}\right)^{(n+m)/2} \Gamma\left(\frac{n+m}{2}\right).
\end{aligned} \tag{8}$$

Another change in variables is performed to obtain $p_{\text{SB}}(x; n, m)$ with $x = f(y) = \sqrt{\frac{1}{1+y}}$.

$$\begin{aligned}
p_{\text{SB}}(x; n, m) &= \left| \frac{d}{dx} \left(f^{-1}(x) \right) \right| \cdot p_Q(f^{-1}(x); \sigma, n, m) \\
&= \left| -2x^{-3} \right| \cdot \frac{k_n k_m}{4} \cdot \left(x^{-2} - 1 \right)^{n/2-1} \cdot \left(\frac{2\sigma^2}{x^{-2}} \right)^{(n+m)/2} \Gamma\left(\frac{n+m}{2}\right) \\
&= \frac{k_n k_m}{2} \cdot \left(2\sigma^2 \right)^{(n+m)/2} \cdot x^{n+m-3} \left(x^{-2} - 1 \right)^{n/2-1} \Gamma\left(\frac{n+m}{2}\right) \\
&= \frac{(2\sigma^2)^{(n+m)/2} \cdot \Gamma\left(\frac{n+m}{2}\right)}{2 \cdot 2^{(n/2)-1} \cdot 2^{(m/2)-1} \cdot \sigma^n \sigma^m \cdot \Gamma\left(\frac{n}{2}\right) \Gamma\left(\frac{m}{2}\right)} \cdot x^{n+m-3} \left(x^{-2} - 1 \right)^{n/2-1} \quad (9) \\
&= \frac{2}{B\left(\frac{n}{2}, \frac{m}{2}\right)} \cdot x^{n+m-3} \left(x^{-2} - 1 \right)^{n/2-1} \\
&= \frac{2}{B\left(\frac{n}{2}, \frac{m}{2}\right)} \cdot x^{m-1} \left(1 - x^2 \right)^{n/2-1} \\
&= \frac{2}{B\left(\frac{n}{2}, \frac{m}{2}\right)} \cdot \left(x^2 \right)^{(m-1)/2} \left(1 - x^2 \right)^{n/2-1}
\end{aligned}$$

□

Corollary 1. *The squared length $|\hat{v}_{1:m}|^2$ of the subvector is distributed according to a Beta distribution with $\alpha = m/2$, $\beta = n/2$.*

Theorem 2. *If X is a random variable distributed according to $X \sim p_{\text{SB}}(x; n, m)$, then the cumulative distribution function is*

$$F_{\text{SB}}(x; n, m) = \frac{B(x^2; \frac{m}{2}, \frac{n}{2})}{B(\frac{m}{2}, \frac{n}{2})}. \quad (10)$$

Proof.

$$\begin{aligned}
F_{\text{SB}}(x; n, m) &= \int_0^x \frac{2}{B\left(\frac{n}{2}, \frac{m}{2}\right)} \cdot \left(x^2 \right)^{(m-1)/2} \left(1 - x^2 \right)^{n/2-1} dx \\
&= \frac{1}{B\left(\frac{n}{2}, \frac{m}{2}\right)} \cdot \int_0^x 2x \cdot x^{-1} \cdot \left(x^2 \right)^{(m-1)/2} \left(1 - x^2 \right)^{n/2-1} dx \quad (11) \\
&= \frac{1}{B\left(\frac{n}{2}, \frac{m}{2}\right)} \cdot \int_0^x \underbrace{2x}_{\phi'(x)} \cdot \underbrace{\left(x^2 \right)^{m/2-1} \left(1 - x^2 \right)^{n/2-1}}_{f(\phi(x))} dx
\end{aligned}$$

Applying a substitution with $\phi(x) = x^2$ gives:

$$\begin{aligned}
F_{\text{SB}}(x; n, m) &= \frac{1}{B\left(\frac{n}{2}, \frac{m}{2}\right)} \cdot \int_{\phi(0)}^{\phi(x)} (\phi(x))^m (1 - \phi(x))^{n/2-1} d\phi(x) \\
&= \frac{B(x^2; \frac{m}{2}, \frac{n}{2})}{B(\frac{m}{2}, \frac{n}{2})} \quad (12)
\end{aligned}$$

□