Text S2

In this section, we define a (pointed) set of concept trees and a partial (approximation) ordering over them, and together they constitute a poset and hence a category. Moreover, a product in this category is a greatest lower bound on the approximation of two concept trees. This categorical approach provides the formal basis for deriving Gentner’s systematicity from universal constructions.

Definition (n-ary relational concept). An n-ary relational concept is a concept that takes n concepts as related arguments. Constants or features are regarded as 0-ary (nullary) relational concepts. Predicates are regarded as 1-ary (unary) relational concepts. Binary relations are regarded as 2-ary, ternary relations as 3-ary, quaternary as 4-ary relational concepts, etc.

Example (n-ary relational concepts). Nouns and values, e.g., ball, red, are typically 0-ary relational concepts, as they do not take any concepts as relational arguments. Categories of features, e.g., colour, are usually 1-ary relational concepts, as they take a single argument as the related concept, e.g., colour red. Loves is a binary relational concept, e.g., John loves Mary. Gives is a ternary relational concept, e.g., John gives Mary a book.

Definition (N-ary relational concept trees). Let $A_n$ be a set of n-ary relational concepts, for $0 \leq n \leq N$. A relational concept tree is a pair consisting of a single n-ary relational concept ($a \in A_n$) leaf and a sequence of n relational concept branch trees, denoted $(t_1, \ldots, t_n)$, or in shorter index form, $(t_i)_{i=1}^n$. A relational concept tree is denoted $\langle a, (t_1, \ldots, t_n) \rangle$, or $\langle a, (t_i)_{i=1}^n \rangle$. When $n = 0$ (a constant concept tree), the sequence of branch trees is empty, denoted $\langle \rangle$, and in a slight abuse of notation, a constant concept tree $\langle a, () \rangle$ is simply denoted $\langle a \rangle$. An N-ary relational concept tree is a relational concept tree with maximum relational concept arity $N$ over all subtrees.

Example (N-ary relational concept trees). An example of a 0-ary (constant) relational concept tree is $\langle \text{John} \rangle$. An example of a 1-ary relational concept tree is $\langle \text{colour}, (\langle \text{red} \rangle) \rangle$. An example of a 2-ary relational concept tree is $\langle \text{loves}, (\langle \text{John} \rangle, \langle \text{Mary} \rangle) \rangle$.

Definition (Relational concept order). The relational concept order of a concept in a concept tree is either: zero for 0-ary relational concepts; or one plus the maximum order of the concepts in its branch trees for $i$-ary concepts, $1 \leq i$. Concepts of order one are called first-order, and concepts of order greater than one are called higher-order relational concepts.

Example (Relational concept order). Loves is a first-order concept in $\langle \text{loves}, (\langle \text{John} \rangle, \langle \text{Mary} \rangle) \rangle$. Knows is a higher(second)-order concept in the tree $\langle \text{knows}, (\langle \text{Sue} \rangle, \langle \text{loves}, (\langle \text{John} \rangle, \langle \text{Mary} \rangle) \rangle) \rangle$. 
**Definition (Pointed tree).** A pointed tree is an $N$-ary relational concept tree constructed from a set of relational concept trees that includes a designated point element $\bot$. The point is the “unknown” concept tree.

**Example (Pointed tree).** An example is $\langle \text{loves}, (\langle \text{John}, \bot \rangle) \rangle$, which indicates that what is loved by John is not known.

**Definition (Approximation ordering).** The approximation ordering $\sqsubseteq$ is a binary relation on a set of $N$-ary relational (approximation) concept trees, $T_\bot$, that is defined for $0 \leq n \leq N$ by:

$$\bot \sqsubseteq t \quad \text{and} \quad \langle a, (t_i)_{i=1}^n \rangle \sqsubseteq \langle b, (r_i)_{i=1}^n \rangle \Leftrightarrow (a = b) \land \left( \bigwedge_{i=1}^n t_i \sqsubseteq r_i \right)$$

**Example (Tree orderings).** The following are examples of approximation orderings on pointed trees: $\bot \sqsubseteq \bot$; $\bot \sqsubseteq (\langle \text{John} \rangle)$; and $\langle \text{loves}, (\langle \text{John}, \bot \rangle) \rangle \sqsubseteq \langle \text{loves}, (\langle \text{John}, \langle \text{Mary} \rangle \rangle) \rangle$. Examples of pairs of trees that are not ordered are: $\langle \text{John} \rangle$ and $\langle \text{Mary} \rangle$; and $\langle \text{loves}, (\langle \text{John}, \bot \rangle) \rangle$ and $\langle \text{loves}, (\bot, \langle \text{Mary} \rangle) \rangle$.

**Remark (Least).** $\bot$ is the least element of $T_\bot$, i.e. $\bot \sqsubseteq t$ for all $t \in T_\bot$.

**Proposition (Approximation partial order).** The approximation ordering $\sqsubseteq$ is a partial order.

**Proof.** We are required to prove that $\sqsubseteq$ is: reflexive, $t \sqsubseteq t$; transitive, $s \sqsubseteq t \land t \sqsubseteq r \Rightarrow s \sqsubseteq r$; and antisymmetric, $t \sqsubseteq r \land r \sqsubseteq t \Rightarrow t = r$ for all $s, t, r \in T_\bot$. The proof is by structural induction. Structural induction proceeds analogously to mathematical induction. First, we prove that the target proposition holds for the base case. Next, we prove that if the target proposition holds for trees $(t_i)_{i=1}^n$, then it holds for $\langle a, (t_i)_{i=1}^n \rangle$.

**Reflexive:**

Base ($t = \bot$): Immediate, since $\bot \sqsubseteq \bot$ (from the definition).

Hypothesis $(\langle t_i \rangle_{i=1}^n)$: For each $i$, we assume $t_i \sqsubseteq t_i$.

Induction $(\langle a, (t_i)_{i=1}^n \rangle)$: Immediate (from the definition), since $a = a$ and $t_i \sqsubseteq t_i$ (by hypothesis) implies $\langle a, (t_i)_{i=1}^n \rangle \sqsubseteq \langle a, (t_i)_{i=1}^n \rangle$.

**Transitive:**
Analogy and cognitive architecture

Base ($s, t, r = \bot$):

\[
s \subseteq t \land t \subseteq r \quad \text{(definition)}
\]
\[
\Rightarrow \bot \subseteq \bot \quad \text{(substitution)}
\]
\[
\Rightarrow s \subseteq r
\]

Hypothesis ($((s_i)_{i=1}^n, (t_i)_{i=1}^n, (r_i)_{i=1}^n)$): For each $i$, we assume the following holds: $s_i \subseteq t_i \land t_i \subseteq r_i \Rightarrow s_i \subseteq r_i$.

Induction ($((a, (s_i)_{i=1}^n), (b, (t_i)_{i=1}^n), (c, (r_i)_{i=1}^n))$):

\[
\langle a, (s_i)_{i=1}^n \rangle \subseteq \langle b, (t_i)_{i=1}^n \rangle \land \langle a, (s_i)_{i=1}^n \rangle \subseteq \langle c, (r_i)_{i=1}^n \rangle \quad \text{(definition)}
\]
\[
\Rightarrow (a = b) \land (s_i \subseteq t_i)_{i=1}^n \land (b = c) \land (t_i \subseteq r_i)_{i=1}^n \quad \text{(=, hypothesis)}
\]
\[
\Rightarrow (a = c) \land (s_i \subseteq r_i)_{i=1}^n \quad \text{(definition)}
\]
\[
\Rightarrow \langle a, (s_i)_{i=1}^n \rangle \subseteq \langle c, (r_i)_{i=1}^n \rangle
\]

Antisymmetric:

Base ($t, r = \bot$): Immediate.

Hypothesis ($((t_i)_{i=1}^n, (r_i)_{i=1}^n)$): For each $i$, we assume the following proposition holds: $t_i \subseteq r_i \land r_i \subseteq t_i \Rightarrow t_i = r_i$.

Induction ($((a, (t_i)_{i=1}^n), (b, (r_i)_{i=1}^n))$):

\[
\langle a, (t_i)_{i=1}^n \rangle \subseteq \langle b, (r_i)_{i=1}^n \rangle \land \langle b, (r_i)_{i=1}^n \rangle \subseteq \langle a, (t_i)_{i=1}^n \rangle \quad \text{(definition)}
\]
\[
\Rightarrow (a = b) \land (t_i \subseteq r_i)_{i=1}^n \land (r_i \subseteq t_i)_{i=1}^n \quad \text{(hypothesis)}
\]
\[
\Rightarrow (a = b) \land (t_i = r_i)_{i=1}^n \quad \text{(equality)}
\]
\[
\Rightarrow \langle a, (t_i)_{i=1}^n \rangle = \langle b, (r_i)_{i=1}^n \rangle
\]

\[\square\]

Proposition (Tree category). A pointed set of $N$-ary relational concept trees ($T_\bot$) together with an approximation ordering ($\subseteq$) is a category.

Proof. Immediate from the fact that ($T_\bot, \subseteq$) is a poset. \[\square\]

Definition (gca). The function (binary operator) $\text{gca} : T_\bot \times T_\bot \rightarrow T_\bot$, which produces the greatest
common approximation of N-ary relational concept trees, is defined for all $t, r \in T_\bot$ and $0 \leq m, n \leq N$ by:

$$gca(t, \bot) = \bot$$

$$gca(\bot, r) = \bot$$

$$gca((a, (t_i)_{i=1}^m), (b, (r_j)_{j=1}^n)) = \bot \quad a \neq b$$

$$gca((a, (t_i)_{i=1}^m), (a, (r_i)_{i=1}^n)) = (a, (gca(t_i, r_i))_{i=1}^n)$$

Example (gca). The gca of $h\text{loves}, (h\text{John}, \bot)$ and $h\text{loves}, (\bot, h\text{Mary})$ is $h\text{loves}, (\bot, \bot)$.

Theorem (gca). The greatest common approximation of trees $t, r \in T_\bot$ is the greatest lower bound of $t$ and $r$.

Proof. By structural induction: we need to show that for all $z, t, r \in T_\bot$ when $z \sqsubseteq t \land z \sqsubseteq r$ we have $z \sqsubseteq gca(t, r)$. In short, we need to prove the following target proposition: $z \sqsubseteq t \land z \sqsubseteq r \Rightarrow z \sqsubseteq gca(t, r)$.

Base ($t = \bot$): The only case of $z$ for which $z \sqsubseteq \bot \land z \sqsubseteq r$ is $z = \bot$, since $\bot$ is the least element of $T_\bot$.

Hence, we have:

$$z \sqsubseteq \bot \land z \sqsubseteq r \Rightarrow z \sqsubseteq gca(t, r) \quad \text{(substitution)}$$

$$\Rightarrow \bot \sqsubseteq gca(t, r) \quad \text{(least element)}$$

$$\Rightarrow \text{True} \quad \text{(reduction)}$$

True

Hypothesis ($((t_i)_{i=1}^n)$): For each $i$, we assume the following proposition holds: $z_i \sqsubseteq t_i \land z_i \sqsubseteq r_i \Rightarrow z_i \sqsubseteq gca(t_i, r_i)$.

Induction ($((a, (t_i)_{i=1}^n))$): By definition (approximation), there are two cases of $z$ for which $z \sqsubseteq (a, (t_i)_{i=1}^n)$:

1. $z = \bot$ and
2. $z = (a, (z_i)_{i=1}^n)$, where $z_i \sqsubseteq t_i$.

Case 1: This case has already been proven in the base case, $t = \bot$.

Case 2: Given that $z = (a, (z_i)_{i=1}^n)$, the target proposition only requires us to consider the trees $r$ where
(a, (z_i)_{i=1}^n) \subseteq r. By definition (approximation), we only need to consider the trees \( r = (a, (r_i)_{i=1}^n) \), where \( z_i \subseteq r_i \). Hence, given \( z = (a, (z_i)_{i=1}^n) \), \( r = (a, (r_i)_{i=1}^n) \) and \( z_i \subseteq r_i \), we are required to prove:

\[
z \subseteq (a, (t_i)_{i=1}^n) \land z \subseteq r \Rightarrow z \subseteq gca((a, (t_i)_{i=1}^n), r)
\]

(substitution)

\[
\Rightarrow (a, (z_i)_{i=1}^n) \subseteq gca((a, (t_i)_{i=1}^n), (a, (r_i)_{i=1}^n))
\]

(definition of gca)

\[
\Rightarrow (a, (z_i)_{i=1}^n) \subseteq (a, (gca(t_i, r_i))_{i=1}^n)
\]

(definition of approximation)

\[
\Rightarrow \bigwedge_{i=1}^n z_i \subseteq gca(t_i, r_i)
\]

(hypothesis)

\[
\Rightarrow \text{True}
\]

(reduction)

\[
\text{True}
\]

\[ \square \]

**Theorem (Tree product, gca).** In the (pointed poset as a) category \((T_\bot, \sqsubseteq)\), a product of trees \( t, r \in T_\bot \) is the greatest common approximation tree of \( t \) and \( r \) together with the two approximation arrows, \( \sqsubseteq_i: t \times r \to t \) and \( \sqsubseteq_r: t \times r \to r \), together denoted \((t \times r, \sqsubseteq_i, \sqsubseteq_r)\).

**Proof.** Immediate from the fact that the greatest common approximation tree is the greatest lower bound, a greatest lower bound is a product in a pointed poset as a category, and \((T_\bot, \sqsubseteq)\), as a pointed poset, is a category. \[ \square \]

**Remark (All products).** The category \((T_\bot, \sqsubseteq)\) is said to have all binary products, since gca (hence, product) is defined for all pairs of trees in \( T_\bot \). \[ \square \]