Appendix-S1. Example of an oriented movement model and its problems

Here we consider the movement model defined by equation (2);
\[ \theta_t = (1 - w)(\alpha - \theta_{t-1}) + \theta_{t-1} + \epsilon_t, \quad (2) \]
in which \( \theta_t \) is the heading direction from time \( t \) to \( t + 1 \), \( \alpha \) is the focal direction, and \( w \) is an unknown parameter.

When \( \theta_{t-1} \) is close to \( \alpha \), this model seems reasonable (Fig. S1A). However, if \( \theta_{t-1} \) is near \( \alpha + \pi \equiv \alpha - \pi \) (mod \( 2\pi \)), equation (2) causes ecologically unreasonable movements; very similar previous directions, \( \psi_{t-1} \) and \( \varphi_{t-1} \), result in almost opposite directions in the subsequent step (B). In other words, equation (2) is not continuous at \( \alpha + \pi \), and the discontinuity becomes clear if equation (2) is graphically displayed as (C). The discontinuity makes the model mathematically not so tractable.

Kato’s circular auto-regressive model resolves this discontinuity by using the sigmoidal curve shown in (C) \( (w = 0.5, \alpha = \pi/3 \) in equation (4)).

Note: More precisely, \( \theta_t \) is given by equation (4) if \( \theta_{t-1} \neq \alpha + \pi \), and \( \theta_t = \alpha + \pi + \epsilon_t \) if \( \theta_{t-1} = \alpha + \pi \). This inconvenience is resolved if the equation is expressed by complex numbers; then, equation (5) is known as the Möbius transformation [S1].

Figure S1. Illustrative explanation on model (2) and its problems. In (C), \( \alpha = \pi/3 \), \( w = 0.5 \). \( \circ \cdots \circ \) is used for showing discontinuity, while continuously connected parts as circular variables are connected by \( \bullet - - - \circ \).

References