Appendix S1 Expectation of the reciprocal of a scaled inverse Chi-square random variable

Given that
\[ \gamma_j | \sigma_j^2 \sim N(0, \sigma_j^2) \]
and
\[ \sigma_j^2 \sim \frac{\nu \gamma S^2}{\lambda^2}, \]

The joint distribution of \( \gamma_j \) and \( \sigma_j^2 \) is
\[ p(\gamma_j, \sigma_j^2) \propto \exp\left(-\frac{\nu \gamma S^2}{2\sigma_j^2}\right) (\sigma_j^2)^{-\left(1+\frac{\nu}{2}\right)} \cdot \exp\left(-\frac{\gamma_j^2}{2\sigma_j^2}\right) \]
\[ \propto \exp\left(-\frac{\nu \gamma S^2 + \gamma_j^2}{2\sigma_j^2}\right) (\sigma_j^2)^{-\left(1+\frac{\nu+1}{2}\right)}. \]

This is the kernel of the conditional distribution of \( \sigma_j^2 \) given \( \gamma_j \), which is a scaled inverse Chi-square distribution with degrees of freedom \( \nu + 1 \) and scale parameter \( \frac{\gamma_j^2 + \nu \gamma S^2}{\nu+1} \).

To show that
\[ \mathbb{E}_{\sigma_j^2 | \gamma, \gamma_j = \gamma^{(k)}} \left( \frac{1}{\sigma_j^2} \right) = \left( \frac{\gamma_j^{(k)}}{\nu \gamma + 1} \right)^{-1}, \]

it suffices to show that the expectation of the reciprocal of a scaled inverse Chi-square variable is the reciprocal of its scale parameter. Suppose \( X \) is a scaled inverse Chi-square random variable with degrees of freedom \( \nu \) and scale parameter \( S^2 \), the probability density function for \( X \) is given by
\[ p(x | \nu, S^2) = \left( \frac{\nu S^2}{\Gamma(\frac{\nu}{2})} \right)^{-\frac{\nu}{2}} \cdot \exp\left(-\frac{\nu S^2}{2x}\right) \cdot \frac{1}{x^{1+\frac{\nu}{2}}}. \]

It follows that the probability density function for \( Y = \frac{1}{X} \) is
\[ p(y | \nu, S^2) = \left( \frac{\nu S^2}{\Gamma(\frac{\nu}{2})} \right)^{-\frac{\nu}{2}} \cdot \exp\left(-\frac{\nu S^2}{2y}\right) \cdot \frac{1}{y^{1+\frac{\nu}{2}}} \cdot \left(-\frac{1}{y^2}\right) \]
\[ = \left( \frac{\nu S^2}{\Gamma(\frac{\nu}{2})} \right)^{-\frac{\nu}{2}} \cdot \exp\left(-\frac{\nu S^2}{2y}\right) \cdot y^{\frac{\nu}{2}-1}. \]

This is the probability density function of Gamma distribution with shape parameter \( \frac{\nu}{2} \) and rate parameter \( \frac{\nu S^2}{2} \). The expectation of Gamma distribution is the shape over rate and therefore the expectation of \( \frac{1}{X} \) is \( \frac{1}{\frac{S^2}{2}} \).