RESEARCH ARTICLE

Integral equation solutions for the average run length for monitoring shifts in the mean of a generalized seasonal ARFIMAX(P, D, Q, r)_s process running on a CUSUM control chart

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Abstract

The CUSUM control chart is suitable for detecting small to moderate parameter shifts for processes involving autocorrelated data. The average run length (ARL) can be used to assess the ability of a CUSUM control chart to detect changes in a long-memory seasonal autoregressive fractionally integrated moving average with exogenous variable (SARFIMAX) process with underlying exponential white noise. Herein, new ARLs via an analytical integral equation (IE) solution as an analytical IE and a numerical IE method to test a CUSUM control chart's ability to detect a wide range of shifts in the mean of a SARFIMAX (P, D, Q, r)_s process with underlying exponential white noise are presented. The analytical IE formulas were derived by using the Fredholm integral equation of the second type while the numerical IE method for the approximate ARL is based on quadrature rules. After applying Banach's fixed-point theorem to guarantee its existence and uniqueness, the precision of the proposed analytical IE ARL was the same as the numerical IE method. The sensitivity and accuracy of the ARLs based on both methods were assessed on a CUSUM control chart running a SARFIMAX(P, D, Q, r)_s process with underlying exponential white noise. The results of an extensive numerical study comprising the examination of a wide variety of out-of-control situations and computational schemes reveal that none of the methods outperformed the IE. Specifically, the computational scheme is easier and can be completed in one step. Hence, it is recommended for use in this situation. An illustrative example based on real data is also provided, the results of which were found to be in accordance with the research results.

Introduction

The discipline of statistical process control (SPC) provides tools such as control charts for monitoring processes and detecting changes in a given in-control model. The Shewhart, cumulative sum (CUSUM), and exponentially weighted moving average (EWMA) control

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charts are the three most commonly used ones. The CUSUM and EWMA control charts can effectively detect small shifts in the parameters of interest (usually location parameters) whereas the Shewhart control chart such as the well-known $\bar{X}$ -chart is used to detect large shifts in process parameters. Details of the charts can be found in Montgomery [1]. The CUSUM control chart was used in the present study due to its superior capability of detecting small parameter shifts and to take advantage of the benefits of using an upper-sided control chart.

Small levels of autocorrelation between successive observations can have a significant effect on the statistical properties of control charts. Numerous researchers have considered the effect of autocorrelation on control chart performance. Johnson and Bagshaw [2] and Bagshaw and Johnson [3] concluded that even small levels of autocorrelation generated by using autoregressive AR(1) or moving average MA(1) time series models made it more difficult to detect out-of-control signals on a conventional CUSUM control chart. Nevertheless, a CUSUM control chart has been used for monitoring shifts in the mean of a stationary ARMA (p, q) model process, and its performance has been evaluated when applied to processes with autocorrelated data [4].

Observations from real-world stochastic processes frequently exhibit a time series component. In particular, economic observations can have AR and MA components in a time series model. The error or so-called white noise, which is defined as the difference between the actual and approximated values, should be kept to a minimum to maintain the highest possible accuracy rate. Although the white noise (also referred to as Gaussian white noise) generated by autocorrelated observations usually follows a normal distribution, this is not always the case. Indeed, non-Gaussian white noise processes are useful for modeling a wide variety of phenomena, including wind speed, and the oxygen concentration and flow rate of water. Numerous studies have been conducted on time series models with non-Gaussian white noise, and when it is exponentially distributed (referred to as exponential white noise) is particularly interesting. Jacob and Lewis [5] considered an ARMA(1,1) process with exponential white noise, while Mohamed and Hocine [6] conducted a Bayesian analysis of an AR(1) process with exponential white noise. Furthermore, Pereira and Turkman [7] conducted a Bayesian analysis of threshold AR models with exponential white noise, while Suparman [8] recently proposed parameter estimations for an AR model of unknown order with exponential white noise.

Time series models are capable of capturing process correlation. An important class of time series model is the stationary process in which it is assumed that the process remains stable around a constant mean. For monitoring autocorrelated processes, the type of model can provide a framework for establishing statistical control. The following basic time series models were used in this study. The conventional Box-Jenkins AR integrated MA (ARIMA) model can be generalized as the AR fractionally integrated MA (ARFIMA) model that allows non-integer (fractional) values for the differencing parameter. This is advantageous when modeling time series with inherent long-range dependence (i.e., long-memory) [9, 10]. Accordingly, ARFIMA models are appropriate for time series long-memory processes, and exogenous variables can be incorporated into the time series model (i.e., ARFIMAX) to improve the latter's performance. Ebens [11] first introduced the ARFIMAX model with k exogenous variables and used it to estimate the realized volatilities in a Dow Jones Industrial Average portfolio, while Degiannakis [12] estimated and forecasted intraday realized volatility using ARFIMAX and ARFIMAX-TARCH (threshold AR conditional heteroskedasticity) models.

Seasonality (seasonal periods over a single year) is a phenomenon that affects time series. Numerous scholars have discussed ARFIMA models with seasonal components, and it has been used to explain a large number of real-world cyclical phenomena in long-memory

processes. For instance, it occurs in series of revenues, inflation rates, monetary aggregates, and gross national product. In the current study, we concentrate on seasonal ARFIMAX (SARFIMAX) models. In terms of economic forecasting and other fields, cases where an exogenous variable is included in the forecasting model are usually more accurate than ones without it. In addition, time series model processes have been monitored by using control charts. For instance, Ramjee [13] discovered that the performances of Shewhart and EWMA control charts with correlated observations in an ARFIMA model were not good, and so proposed the hyperbolic weighted MA (HWMA) control chart instead. Subsequently, Ramjee et al. [14] designed an HWMA forecast-based control chart particularly designed for a non-stationary ARFIMA model with autocorrelated data. When residual control charts with ARFIMA and ARIMA models were used to monitor the air quality Taiwan [15], the one with the ARFIMA model was more appropriate than the one with the ARIMA model. Rabyk and Schmid [16] recently introduced the EWMA control chart for detecting changes in the mean of a long-memory process with the control chart's design based on an ARFIMA(p, d, q) process.

The most widespread measure of evaluating the performance of the control chart is the average run length (ARL). The power of a control chart is defined as the probability of its out-of-control signal detection, while the average number of samples required to signal an out-of-control situation in the process (or expected value of the run length) is represented by the ARL. The in-control and out-of-control ARLs are denoted as ARL_0 and ARL_1 , respectively: ARL_0 should be as large as possible whereas ARL_1 should be as small as possible when the process changes undesirably. To improve the out-of-control detection performance on a CUSUM control chart, two alternative schemes are proposed that help to minimize ARL_1 when ARL_0 is fixed.

Assessing the ARL of a control chart using randomness and numerical methods is possible using a variety of techniques, including Monte Carlo, Markov Chain, and integral equations (IEs). The Monte Carlo simulation technique can be used to determine the expected run length until the target ARL value approaches that of an IE under the assumption of a sufficiently long simulation time. For the Markov Chain approach, the range between the upper and lower control limits is divided into sub-intervals, while the probabilities of the runs on the control chart are presented as a transitional probability matrix [17] over the sub-interval so that the ARL can be evaluated by inverting the matrix. In the meantime, the IE method comprises two alternatives: analytical IE and numerical IE. The analytical IE method generates exact ARLs or explicit formulas. The numerical IE method is transformed into a system of linear equations that can be solved for the ARL by using certain linear algebra-based methods. Crowder [18] provided a numerical method by using an IE of the second kind and a computer program for ARL evaluation of a normally distributed process on a two-sided EWMA control chart. Moreover, Hawkins [19] solved IEs for determining the ARL of a CUSUM control chart. Later, ARLs by using analytical IEs and the Markov Chain approach were used on EWMA and CUSUM control charts for observations from an AR(1) process with additional random error [20]. To evaluate the ARL for a SARMA(1, 1)_s process with exponential white noise on a CUSUM control chart, Phanyaem [21] developed explicit formulas for IEs initially based on the SARMA model. In addition, analytical IEs for the solution for the ARL for a long-memory SARFIMA model on CUSUM control chart were presented by Peerajit et al. [22]. Recently, complex data from seasonal and non-seasonal MA processes with exogenous variables were used to evaluate the ARL on a CUSUM control chart by Sunthornwat and Areepong [23]. Moreover, it is important to apply Banach's fixed-point theorem to prove the existence and uniqueness of an ARL. In this paper, our focus was on establishing ARLs for monitoring changes in a long-memory SARFIMAX process with underlying exponential white noise running on a CUSUM control chart.

The remainder of the article is organized as follows. Section 2 contains a brief review of the ARFIMAX models taking long memory and seasonality into account and descriptions of the CUSUM control chart, as well as the characteristics of the ARL and Banach's fixed-point theorem. Theoretical proof for the existence and uniqueness of analytical IE, along with its comparable accuracy to the numerical ARL are covered in Section 3. The next section comprises derivations of the proposed analytical and numerical IE methods for the ARL for monitoring changes in a long-memory SARFIMAX process with underlying exponential white noise on a CUSUM control chart. The results of an evaluation of the performances of the IE methods are provided in Section 5. In Section 6, application of the methods to the monthly stock price data for PTT Public Company Ltd with the EUR/THB exchange rate as the exogenous variable is reported. Conclusions and a discussion of possible future work are provided in Section 7. Finally, the appendix contains technical details of the computation of the ARL by using the proposed methods.

Preliminaries

The relevant fundamentals of the long-memory SARFIMAX process with underlying exponential white noise are described in this section. In addition, we determine whether detecting changes in the mean of a generalized SARFIMAX process running on a CUSUM control chart is suitable for this study. For simplicity, the second subsection is focused on the upper-sided CUSUM control chart only. Next, performance evaluation of a control chart and the characteristics of the ARL are covered. In the last subsection, definitions related to Banach's fixed-point theorem are covered.

Long-memory SARFIMAX processes with underlying exponential white noise

The ARFIMA model, a generalized form of the usual Box-Jenkins ARIMA model but allowing non-integer (fractional) values for the differencing parameter, is useful for modeling time series with inherent long-range dependence [9, 10]. Furthermore, including an exogenous variable in the ARFIMA model produces the ARFIMAX model with better performance [12]. In practice, time series are often found in meteorology, economics, hydrology, and astronomy [24]. Seasonality is an autocorrelation structure where the data vary according to specific periods in a year, a phenomenon that is included in our analysis.

A SARFIMAX model of order $(P, D, Q, r)_s$, with AR order (P), fractional integration order (D), MA order (Q), and exogenous variables order (r), and period (s) for seasonal data is an extension of the ARFIMAX(p, d, q, r) model applied to a seasonal time series. SARFIMAX(P, D, Q, r)_s is stationary and invertible for $D \in (-1, 0.5)$; for $D > 0$, where D is seasonal fractionally differentiated/ integration and $D \in (0, 0.5)$ is for a long-memory process on a CUSUM control chart.

Definition 1. Let $\{Y_t\}_{t \in \mathbb{Z}}$ be a sequence from a SARFIMAX(P, D, Q, r)_s process defined as

$$\Phi_p(B^s)(1 - B^s)^D Y_t = \mu + \Theta_Q(B^s)\xi_t + \sum_{l=1}^r \omega_l X_{lt}, \quad \xi_t \sim \text{Exp}(\beta). \quad (1)$$

where μ is the constant of the process, $X_{lt}, l = 1, 2, \dots, r$ are exogenous variables at time t , The $\omega_l, l = 1, 2, \dots, r$ are coefficients corresponding to r exogenous variables. It is assumed that the error terms (ξ_t) are white noise, i.e., $\xi_t \sim \text{Exp}(\beta)$. The operator B is a backward-shift operator (i.e., $B^s Y_t = Y_{t-s}$), $(1 - B^s)^D$ is seasonal fractional difference operator, where s refers to the number of seasonal periods per year (e.g., $s = 12$ for monthly data), and $\Phi_p(B^s)$ and $\Theta_Q(B^s)$ are

the coefficient of seasonal AR and MA polynomials in B order P and Q ; $s \in \mathbb{N}$, respectively, defined by

$$\Phi_p(B^s) = 1 - \Phi_1 B^s - \Phi_2 B^{2s} - \dots - \Phi_p B^{Ps} = 1 - \sum_{i=1}^P \Phi_i B^{is}$$

and

$$\Theta_Q(B^s) = 1 - \Theta_1 B^s - \Theta_2 B^{2s} - \dots - \Theta_Q B^{Qs} = 1 - \sum_{j=1}^Q \Theta_j B^{js},$$

where the coefficient of seasonal Φ_i , $i = 1, 2, \dots, P$, and Θ_j , $j = 1, 2, \dots, Q$ are real numbers.

Remark: For all real D , seasonal fractional difference operator $(1 - B^s)^D$ with seasonality $s \in \mathbb{N}$ is defined by employing a binomial expansion as follows:

$$(1 - B^s)^D = \sum_{k \geq 0} \binom{D}{k} (-1)^k B^{ks} = 1 - DB^s + \frac{D(D-1)}{2!} B^{2s} - \frac{D(D-1)(D-2)}{3!} B^{3s} + \dots,$$

where D becomes crucial for describing the degree of seasonal persistence.

The result of the process is based on binomial expansion. After simplification of the derivation of process, the generalized SARFIMAX(P, D, Q, r)_s process is as follows:

$$\begin{aligned} Y_t = & \mu + \xi_t - \sum_{j=1}^Q \Theta_j \xi_{t-js} + \sum_{l=1}^r \omega_l X_{lt} + \left(DY_{t-s} - \frac{D(D-1)}{2!} Y_{t-2s} + \frac{D(D-1)(D-2)}{3!} Y_{t-3s} - \dots \right) \\ & + \sum_{i=1}^P \Phi_i \left(Y_{t-is} - DY_{t-(i+1)s} + \frac{D(D-1)}{2!} Y_{t-(i+2)s} - \frac{D(D-1)(D-2)}{3!} Y_{t-(i+3)s} + \dots \right), \end{aligned} \quad (2)$$

where initial value is normally the process mean, seasonal AR, seasonal MA and exogenous variables coefficient are, respectively, $\xi_{t-s}, \xi_{t-2s}, \dots, \xi_{t-Qs} = 1$, $-1 \leq \Phi_i \leq 1$, $-1 \leq \Theta_j \leq 1$, and $-1 \leq \omega_l \leq 1$. It is assumed that the initial value of generalized SARFIMAX(P, D, Q, r)_s process $Y_{t-s}, Y_{t-2s}, \dots, Y_{t-Ps}, Y_{t-(P+1)s}, \dots = 1$. In this paper, Y_t is long memory process which is assess $D \in (0, 0.5)$.

Process monitoring on a CUSUM and EWMA control charts. One of the simplest and efficient SPC techniques for detecting small and moderate changes in the mean of a process is the CUSUM and EWMA control charts. The statistic of the upper-sided CUSUM control chart [25] can be expressed by the following recursive equation:

$$C_t = \max \{0, C_{t-1} + Y_t - \eta\}, \text{ for } t = 1, 2, \dots, \quad (3)$$

where CUSUM control chart parameter Y_t is a sequence of long-memory SARFIMAX(P, D, Q, r)_s process with underlying exponential white noise. $\eta > 0$ and $C_0 \geq 0$ as the reference and starting value, respectively. The starting value of the CUSUM statistic is φ (i.e., $C_0 = \varphi$, φ is the initial value; $\varphi \in [0, b]$), and parameter η is used to balance the series (C_t) that is commonly chosen to be larger than but close to the value of the in-control mean.

Remark: If C_t control limit $b > 0$, an alarm is triggered that indicates that the process might be out-of-control. In view of an exact performance evaluation, we chose rational or even real number design parameter η .

For the above CUSUM sequence (C_t), the corresponding stopping time with predetermined threshold b can be deployed:

$$\tau_b = \inf \{t > 0 ; C_t > b\}, \text{ for } \varphi \leq b, \quad (4)$$

where τ_b is the stopping time, b is the upper control limit (UCL) of CUSUM chart.

The statistic of the upper-sided EWMA control chart [26] can be expressed by the following recursive equation:

$$Z_t = (1 - \lambda)Z_{t-1} + \lambda Y_t, \text{ for } t = 1, 2, \dots,$$

where λ is an exponential smoothing parameter with $0 < \lambda \leq 1$. The starting value of the EWMA statistic is $Z_0 = Y_0$, the target mean value is μ , Y_t is a processes with mean μ and variance σ^2 .

The upper control limit (UCL) and lower control limit (LCL) to detect the sequence is given by,

$$\begin{aligned} UCL &= \mu_0 + L\sigma \sqrt{\frac{\lambda}{2-\lambda} [1 - (1-\lambda)^{2t}]}, \\ LCL &= \mu_0 - L\sigma \sqrt{\frac{\lambda}{2-\lambda} [1 - (1-\lambda)^{2t}]}, \end{aligned} \quad (5)$$

where L is the width of the control limits, where μ_0 is the target mean, σ is the process standard deviation.

Characteristics of the ARL. Throughout the whole paper, the error term (ξ_t) is a sequence of continuous i.i.d. random variables taken from an exponential distribution with distribution function. ($F(x, \beta)$), which is used in the next section concerning monitoring of the mean. We consider the following simplified change-point models:

$$\xi_t \sim \begin{cases} \text{Exp}(\beta_0), & t = 1, 2, \dots, v-1 \\ \text{Exp}(\beta_1), & t = v, v+1, \dots \end{cases} \quad (6)$$

where β_0 and β_1 are known parameters; $\beta_1 > \beta_0$. As mentioned previously, when considering the change point in Eq (6), the ARL defined with $E_v(\cdot)$ can be characterized as follows:

$$\text{ARL} = \begin{cases} \text{ARL}_0 = E_\infty(\tau_b), & v = \infty \text{ (no change)} \\ \text{ARL}_1 = E_1(\tau_b), & v = 1 \text{ (change)} \end{cases},$$

where $E_v(\cdot)$ denotes the expectation under distribution $F(x, \beta)$ for a given change-point time (v). $v = \infty$ indicates no change in the statistical process and is referred to as the in-control ARL (ARL_0). Herein, $v = 1$ marks the first time point that a change takes place from β_0 to β in the statistical process, which is called out-of-control ARL (ARL_1).

Definitions and theories for evaluating the ARL. Definitions, theories, and concepts necessary to prove the existence and uniqueness of the IEs in the proof based on the functional analysis are covered in this section.

Definition 2. Let (M, d) be a metric space and let $\Upsilon: M \rightarrow M$ be a mapping. A point $u \in M$ is called a fixed point of Υ if $\Upsilon(u) = u$.

Remark: Fixed point theorems guarantee the existence of a fixed point under appropriate conditions on the map Υ and the set M .

Definition 3. A normed vector space M is a Banach space if the metric space (M, d) is complete, where $d(u, v) = \|u - v\|$, for all $u, v \in M$.

Consider the Banach space is n -dimensional Euclidean space $\mathbb{R}^n$, where the norm $|\cdot|$ is given by the Euclidean distance. The $C(M)$ is space of continuous functions where M is a compact interval and the norm $\|\cdot\|$ is given by $\|f\|_\infty = \sup_{u \in M} |f(u)|$ for $f \in C(M)$.

Definition 4. Let (M, d) be a metric space and let $\Upsilon: M \rightarrow M$ be a map, and Υ is called contraction if there exists a fixed constant $\rho \in [0, 1)$ such that

$$d(\Upsilon(u), \Upsilon(v)) \leq \rho d(u, v), \text{ for all } u, v \in M.$$

Theorem 1 (Banach's fixed-point theorem or the contraction theorem) [27]. Let (M, d) be a complete metric space, then each contraction map $\Upsilon: M \rightarrow M$ has a unique fixed point.

The existence and uniqueness of the analytical IE for the ARL on a CUSUM control chart

The analytical IE for the ARL of a CUSUM control chart running a long-memory SARFIMAX process corresponds to a Fredholm integral equation of the second type. In this section, Banach's fixed-point theorem is applied to prove the existence and uniqueness of the analytical IE solution and demonstrate that it has the same accuracy as the numerical IE method.

Throughout, this paper, let $\ell(\varphi)$ be the ARL conditioned on the initial value for a CUSUM control chart running a long-memory SARFIMAX process with underlying exponential white noise [17]. The initial value for monitoring CUSUM statistic $C_0 = \varphi$ is determined at $\varphi \in [0, b]$. Let $\mathbb{P}_\varphi$ and $\mathbb{E}_\varphi$ be the probability measure and induced expectation corresponding to φ . Moreover, ARL is defined as a function of $\ell(\varphi) = \mathbb{E}_\varphi(\tau_b) < \infty$, where $\tau_b = \inf\{t > 0; C_t > b\}$ satisfies the solution for the analytical IE as follows:

$$\ell(\varphi) = 1 + \mathbb{P}_\varphi\{C_1 = 0\}\ell(0) + \mathbb{E}_\varphi[I\{0 < C_1 < b\}\ell(C_1)], \quad (7)$$

$$\text{where indicator function is } I\{0 < C_1 < b\} = \begin{cases} 1; & 0 < C_1 < b \\ 0; & \text{Otherwise.} \end{cases}$$

The IE for the ARL derived from the Fredholm integral equation of the second kind can be written as follows:

$$\ell(\varphi) = 1 + \ell(0)F(\eta - \varphi - Y_t) + \int_0^b \ell(g)f(g + \eta - \varphi - Y_t)dg, \quad (8)$$

where $f(\cdot)$ is the probability density function (pdf) of an exponential distribution [17].

Hence, deriving the analytical IE via Eq (8) becomes

$$\begin{aligned} \ell(\varphi) = & 1 + (1 - \exp \left\{ -\beta \left(\eta - \varphi - \mu - \xi_t + \sum_{j=1}^Q \Theta_j \xi_{t-j} - \sum_{l=1}^r \omega_l X_{lt} - \left(DY_{t-s} - \frac{D(D-1)}{2!} Y_{t-2s} + \frac{D(D-1)(D-2)}{3!} Y_{t-3s} - \dots \right) \right) \right. \\ & \left. - \sum_{i=1}^P \Phi_i \left(Y_{t-is} - DY_{t-(i+1)s} + \frac{D(D-1)}{2!} Y_{t-(i+2)s} - \frac{D(D-1)(D-2)}{3!} Y_{t-(i+3)s} + \dots \right) \right\}) \ell(0) \\ & + \beta (\exp \left\{ \beta \left(\varphi - \eta + \mu + \xi_t - \sum_{j=1}^Q \Theta_j \xi_{t-j} + \sum_{l=1}^r \omega_l X_{lt} + \left(DY_{t-s} - \frac{D(D-1)}{2!} Y_{t-2s} + \frac{D(D-1)(D-2)}{3!} Y_{t-3s} - \dots \right) \right) \right. \\ & \left. + \sum_{i=1}^P \Phi_i \left(Y_{t-is} - DY_{t-(i+1)s} + \frac{D(D-1)}{2!} Y_{t-(i+2)s} - \frac{D(D-1)(D-2)}{3!} Y_{t-(i+3)s} + \dots \right) \right\}) \int_0^b \ell(g) \exp \{-\beta g\} dg, \quad \varphi \in [0, b], \end{aligned} \quad (9)$$

where $\ell(\cdot)$ is an arbitrary function on $[0, b]$. The right-hand side of Eq (9) becomes a

continuous function, and so theoretically, it is clear that the ARL function $\ell(\cdot)$ is a continuous solution on $[0, b]$. for the IE in Eq (9). After that, definitions and Banach's fixed-point theorem are applied to guarantee the existence and uniqueness of the solution for functional $\ell(\varphi)$.

Theorem 2. The existence and uniqueness of the solution for the analytical IE are derived for a long-memory SARFIMAX(P, D, Q, r)_s process with underlying exponential white noise.

Proof: (Existence). As shown in Appendix A in S1 File,

$$\Upsilon(\ell) = \lim_{n \rightarrow \infty} \Upsilon(\ell_n) = \lim_{n \rightarrow \infty} \ell_{n+1} = \ell.$$

In this situation we have

$$\begin{aligned} \Upsilon(\ell(\varphi)) &= 1 + (1 - \exp \left\{ -\beta \left(\eta - \varphi - \mu - \xi_t + \sum_{j=1}^Q \Theta_j \xi_{t-j} - \sum_{l=1}^r \omega_l X_{lt} - \left(DY_{t-s} - \frac{D(D-1)}{2!} Y_{t-2s} + \frac{D(D-1)(D-2)}{3!} Y_{t-3s} - \dots \right) \right) \right. \\ &\quad \left. - \sum_{i=1}^P \Phi_i \left(Y_{t-is} - DY_{t-(i+1)s} + \frac{D(D-1)}{2!} Y_{t-(i+2)s} - \frac{D(D-1)(D-2)}{3!} Y_{t-(i+3)s} + \dots \right) \right\}) \ell(0) \\ &\quad + \beta(\exp \left\{ \beta(\varphi - \eta + \mu + \xi_t - \sum_{j=1}^Q \Theta_j \xi_{t-j} + \sum_{l=1}^r \omega_l X_{lt} + \left(DY_{t-s} - \frac{D(D-1)}{2!} Y_{t-2s} + \frac{D(D-1)(D-2)}{3!} Y_{t-3s} - \dots \right) \right. \\ &\quad \left. + \sum_{i=1}^P \Phi_i \left(Y_{t-is} - DY_{t-(i+1)s} + \frac{D(D-1)}{2!} Y_{t-(i+2)s} - \frac{D(D-1)(D-2)}{3!} Y_{t-(i+3)s} + \dots \right) \right\}) \int_0^b \ell(g) \exp \{-\beta g\} dg, \end{aligned}$$

where $\ell(\varphi)$ is a fixed point of Υ . Hence, the IE in Eq (9) can be rewritten as $\Upsilon(\ell(\varphi)) = \ell(\varphi)$, which is Banach's fixed-point theorem. If operator Υ is a contraction, then fixed point $\Upsilon(\ell(\varphi)) = \ell(\varphi)$.

Proof: (Uniqueness). Let Υ be a contraction mapping on a complete metric space. By considering Banach's fixed-point theorem, it can be shown that $\Upsilon: C([0, b]) \rightarrow C([0, b])$ is a contraction mapping on the complete metric space $(C([0, b]), \|\cdot\|_\infty)$ where metric $C([0, b])$ is the space of all continuous functions $\ell(\varphi)$ on interval; $[0, b]$ endowed with supremum norm $\|\cdot\|_\infty = \sup_{\varphi \in [0, b]} |\ell(\varphi)|$, for every function $\ell(\cdot) \in C([0, b])$. It has a unique solution, as shown in Appendix B in S1 File.

Thus, as proved and guaranteed by applying Banach's fixed-point theorem, the analytical IE for a long-memory SARFIMAX(P, D, Q, r)_s process with underlying exponential white noise on the CUSUM control chart exists and is unique for solutions of functional $\ell(\varphi)$.

Analytical and approximate ARLs for a long-memory SARFIMAX process with underlying exponential white noise on a CUSUM control chart

In this section, we apply the proposed analytical and numerical IEs corresponding to a Fredholm integral equation of the second type representing analytical and approximate ARLs for monitoring changes in the process mean of a long-memory SARFIMAX process with underlying exponential white noise on a CUSUM control chart.

The proposed analytical IE for a SARFIMAX(P, D, Q, r)_s process with underlying white noise on a CUSUM control chart

Theorem 3. Exact formula $\ell(\varphi)$ for monitoring changes in the process mean of a long-memory SARFIMAX process with underlying exponential white noise on a CUSUM control charts can

be expressed as

$$\ell(\varphi) = \exp\{\beta b\} \left((1 + \exp \left\{ \beta \left(\eta - \mu - \zeta_t + \sum_{j=1}^Q \Theta_j \zeta_{t-j} - \sum_{l=1}^r \omega_l X_{lt} - \left(DY_{t-s} - \frac{D(D-1)}{2!} Y_{t-2s} + \frac{D(D-1)(D-2)}{3!} Y_{t-3s} - \dots \right) \right) \right\} - \beta b \right) \right. \\ \left. - \sum_{i=1}^p \Phi_i \left(Y_{t-is} - DY_{t-(i+1)s} + \frac{D(D-1)}{2!} Y_{t-(i+2)s} - \frac{D(D-1)(D-2)}{3!} Y_{t-(i+3)s} + \dots \right) \right) \\ - \exp\{\beta \varphi\}, \text{ for } \varphi \geq 0.$$

Proof: Let $\ell(\varphi)$ be the ARL of the IE corresponding to the long-memory SARFIMAX(P, D, Q, r)_s process on a CUSUM control chart. Thus, $c = \int_0^b \ell(g) \exp\{-\beta g\} dg$ can be substituted as follows:

$$\ell(\varphi) = 1 + (1 - \exp \left\{ -\beta \left(\eta - \varphi - \mu - \zeta_t + \sum_{j=1}^Q \Theta_j \zeta_{t-j} - \sum_{l=1}^r \omega_l X_{lt} - \left(DY_{t-s} - \frac{D(D-1)}{2!} Y_{t-2s} + \frac{D(D-1)(D-2)}{3!} Y_{t-3s} - \dots \right) \right) \right\} \right) \ell(0) \\ + c\beta \left(\exp \left\{ \beta \left(\varphi - \eta + \mu + \zeta_t - \sum_{j=1}^Q \Theta_j \zeta_{t-j} + \sum_{l=1}^r \omega_l X_{lt} + \left(DY_{t-s} - \frac{D(D-1)}{2!} Y_{t-2s} + \frac{D(D-1)(D-2)}{3!} Y_{t-3s} - \dots \right) \right) \right\} \right. \\ \left. + \sum_{i=1}^p \Phi_i \left(Y_{t-is} - DY_{t-(i+1)s} + \frac{D(D-1)}{2!} Y_{t-(i+2)s} - \frac{D(D-1)(D-2)}{3!} Y_{t-(i+3)s} + \dots \right) \right) \right). \quad (10)$$

By replacing $\varphi = 0$ in Eq (10), we obtain

$$\ell(\varphi) = 1 + (1 - \exp \left\{ -\beta \left(\eta - \mu - \zeta_t + \sum_{j=1}^Q \Theta_j \zeta_{t-j} - \sum_{l=1}^r \omega_l X_{lt} - \left(DY_{t-s} - \frac{D(D-1)}{2!} Y_{t-2s} + \frac{D(D-1)(D-2)}{3!} Y_{t-3s} - \dots \right) \right) \right\} \right) \ell(0) \\ + c\beta \left(\exp \left\{ \beta \left(\mu - \eta + \zeta_t - \sum_{j=1}^Q \Theta_j \zeta_{t-j} + \sum_{l=1}^r \omega_l X_{lt} + \left(DY_{t-s} - \frac{D(D-1)}{2!} Y_{t-2s} + \frac{D(D-1)(D-2)}{3!} Y_{t-3s} - \dots \right) \right) \right\} \right. \\ \left. + \sum_{i=1}^p \Phi_i \left(Y_{t-is} - DY_{t-(i+1)s} + \frac{D(D-1)}{2!} Y_{t-(i+2)s} - \frac{D(D-1)(D-2)}{3!} Y_{t-(i+3)s} + \dots \right) \right) \right).$$

$$\therefore \ell(0) = c\beta + \exp \left\{ \beta \left(\eta - \mu - \zeta_t + \sum_{j=1}^Q \Theta_j \zeta_{t-j} - \sum_{l=1}^r \omega_l X_{lt} - \left(DY_{t-s} - \frac{D(D-1)}{2!} Y_{t-2s} + \frac{D(D-1)(D-2)}{3!} Y_{t-3s} - \dots \right) \right) \right\} \\ - \sum_{i=1}^p \Phi_i \left(Y_{t-is} - DY_{t-(i+1)s} + \frac{D(D-1)}{2!} Y_{t-(i+2)s} - \frac{D(D-1)(D-2)}{3!} Y_{t-(i+3)s} + \dots \right) \quad (11)$$

Subsequently, by substituting $\ell(0)$ into Eq (10), it follows that

$$\ell(\varphi) = 1 + c\beta + \exp \left\{ \beta \left(\eta - \mu - \zeta_t + \sum_{j=1}^Q \Theta_j \zeta_{t-j} - \sum_{l=1}^r \omega_l X_{lt} - \left(DY_{t-s} - \frac{D(D-1)}{2!} Y_{t-2s} + \frac{D(D-1)(D-2)}{3!} Y_{t-3s} - \dots \right) \right) \right. \\ \left. - \sum_{i=1}^P \Phi_i \left(Y_{t-is} - DY_{t-(i+1)s} + \frac{D(D-1)}{2!} Y_{t-(i+2)s} - \frac{D(D-1)(D-2)}{3!} Y_{t-(i+3)s} + \dots \right) \right\} - \exp \{ \beta \varphi \}. \quad (12)$$

Constant $c = \int_0^b \ell(g) \exp \{ -\beta g \} dg$, can be derived as

$$c = (1 + c\beta + \exp \left\{ \beta \left(\eta - \mu - \zeta_t + \sum_{j=1}^Q \Theta_j \zeta_{t-j} - \sum_{l=1}^r \omega_l X_{lt} - \left(DY_{t-s} - \frac{D(D-1)}{2!} Y_{t-2s} + \frac{D(D-1)(D-2)}{3!} Y_{t-3s} - \dots \right) \right) \right. \\ \left. - \sum_{i=1}^P \Phi_i \left(Y_{t-is} - DY_{t-(i+1)s} + \frac{D(D-1)}{2!} Y_{t-(i+2)s} - \frac{D(D-1)(D-2)}{3!} Y_{t-(i+3)s} + \dots \right) \right\} \\ \int_0^b \exp \{ -\beta g \} dg - \int_0^b \exp \{ \beta g - \beta g \} dg$$

After rearranging, we obtain

$$\therefore c = \beta^{-1} \exp \{ \beta b \} (1 + \exp \left\{ \beta \left(\eta - \mu - \zeta_t + \sum_{j=1}^Q \Theta_j \zeta_{t-j} - \sum_{l=1}^r \omega_l X_{lt} - \left(DY_{t-s} - \frac{D(D-1)}{2!} Y_{t-2s} + \frac{D(D-1)(D-2)}{3!} Y_{t-3s} - \dots \right) \right) \right. \\ \left. - \sum_{i=1}^P \Phi_i \left(Y_{t-is} - DY_{t-(i+1)s} + \frac{D(D-1)}{2!} Y_{t-(i+2)s} - \frac{D(D-1)(D-2)}{3!} Y_{t-(i+3)s} + \dots \right) \right\} \\ \times (1 - \exp \{ -\beta b \}) - b \exp \{ \beta b \}. \quad (13)$$

Finally, by substituting constant c from Eq (13) into Eq (12), we obtain

$$\ell(\varphi) = 1 + (\exp \{ \beta b \} (1 - \exp \{ -\beta b \}) (1 + \exp \left\{ \beta \left(\eta - \mu - \zeta_t + \sum_{j=1}^Q \Theta_j \zeta_{t-j} - \sum_{l=1}^r \omega_l X_{lt} - \left(DY_{t-s} - \frac{D(D-1)}{2!} Y_{t-2s} + \frac{D(D-1)(D-2)}{3!} Y_{t-3s} - \dots \right) \right) \right. \\ \left. - \sum_{i=1}^P \Phi_i \left(Y_{t-is} - DY_{t-(i+1)s} + \frac{D(D-1)}{2!} Y_{t-(i+2)s} - \frac{D(D-1)(D-2)}{3!} Y_{t-(i+3)s} + \dots \right) \right\} \\ - \beta b \exp \{ \beta b \})$$

$$\ell(\varphi) = \exp \{ \beta b \} (1 + \exp \left\{ \beta \left(\eta - \mu - \zeta_t + \sum_{j=1}^Q \Theta_j \zeta_{t-j} - \sum_{l=1}^r \omega_l X_{lt} - \left(DY_{t-s} - \frac{D(D-1)}{2!} Y_{t-2s} + \frac{D(D-1)(D-2)}{3!} Y_{t-3s} - \dots \right) \right) \right. \\ \left. - \sum_{i=1}^P \Phi_i \left(Y_{t-is} - DY_{t-(i+1)s} + \frac{D(D-1)}{2!} Y_{t-(i+2)s} - \frac{D(D-1)(D-2)}{3!} Y_{t-(i+3)s} + \dots \right) \right\} - \beta b) \\ - \exp \{ \beta \varphi \}; \quad \varphi \geq 0. \quad (14)$$

The proof is complete.

As shown in the previous equation, the in-control ARL (ARL_0) is assigned exponential parameter ($\beta = \beta_0$), as follows:

$$ARL_0 = \exp \{ \beta_0 b \} (1 + \exp \left\{ \beta_0 \left(\eta - \mu - \xi_t + \sum_{j=1}^Q \Theta_j \xi_{t-j} - \sum_{l=1}^r \omega_l X_{lt} - \left(DY_{t-s} - \frac{D(D-1)}{2!} Y_{t-2s} + \frac{D(D-1)(D-2)}{3!} Y_{t-3s} - \dots \right) \right) - \sum_{i=1}^P \Phi_i \left(Y_{t-is} - DY_{t-(i+1)s} + \frac{D(D-1)}{2!} Y_{t-(i+2)s} - \frac{D(D-1)(D-2)}{3!} Y_{t-(i+3)s} + \dots \right) \right\} - \beta_0 b \} - \exp \{ \beta_0 \varphi \}. \quad (15)$$

Similarly, the out-of-control ARL (ARL_1) is assigned exponential parameter ($\beta = \beta_1$), where $\beta_1 = \beta_0(1 + \delta)$, and the shift in the mean is determined by δ and $\beta_0 = 1$ as follows:

$$ARL_1 = \exp \{ \beta_1 b \} (1 + \exp \left\{ \beta_1 \left(\eta - \mu - \xi_t + \sum_{j=1}^Q \Theta_j \xi_{t-j} - \sum_{l=1}^r \omega_l X_{lt} - \left(DY_{t-s} - \frac{D(D-1)}{2!} Y_{t-2s} + \frac{D(D-1)(D-2)}{3!} Y_{t-3s} - \dots \right) \right) - \sum_{i=1}^P \Phi_i \left(Y_{t-is} - DY_{t-(i+1)s} + \frac{D(D-1)}{2!} Y_{t-(i+2)s} - \frac{D(D-1)(D-2)}{3!} Y_{t-(i+3)s} + \dots \right) \right\} - \beta_1 b \} - \exp \{ \beta_1 \varphi \}. \quad (16)$$

These circumstances show that the computational scheme can be completed in one stage.

The numerical ARL for a SARFIMAX(P, D, Q, r), process with underlying white noise on a CUSUM control chart

An approximate ARL using the numerical IE method via Fredholm's integral equation of the second kind was previously formulated for a CUSUM control chart by Peerajit et al. [22]. For this method, an approximate ARL was developed by using the Gauss-Legendre quadrature rules technique.

Definition 5. The quadrature rules usually applied to integral $\int_0^b f(g)dg$ can be approximated by the sum of the areas of a rectangle is as follows:

$$\int_0^b W(g)f(g)dg \approx \sum_{j=1}^m w_j f(v_j), \text{ with } v_j = \frac{b}{m} \left(j - \frac{1}{2} \right), \quad j = 1, 2, \dots, m,$$

where the value of integral f is chosen by applying base b/m with heights at the midpoints of the intervals of length b/m beginning at zero and $W(g)$ is a weight function. Division points $v_1 \leq \dots \leq v_m$ fall within interval $[0, b]$ and $w_j = b/m \geq 0$ set of constant weights $w_j, j = 1, 2, \dots, m$.

Theorem 4. The numerical IE method ($\vartheta(\varphi)$) for monitoring changes in the process mean of a long-memory SARFIMAX process with underlying exponential white noise (Y_t) on a CUSUM control chart can be expressed as

$$\vartheta(\varphi) = 1 + \vartheta(v_1)F(\eta - \varphi - Y_t) + \sum_{j=1}^m w_j \vartheta(v_j)f(v_j + \eta - \varphi - Y_t),$$

with $w_j = \frac{b}{m}$, and $v_j = \frac{b}{m} \left(j - \frac{1}{2} \right)$; $j = 1, 2, \dots, m$.

Proof: By applying $\vartheta(\varphi)$ as the approximate ARL from the IE solution in Eq (9) using the Gauss-Legendre quadrature rule technique and replacing φ with v_j , we obtain

$$\vartheta(v_i) = 1 + \vartheta(v_1)F(\eta - v_i - Y_t) + \sum_{j=1}^m w_j \vartheta(v_j) f(v_j + \eta - v_i - Y_t), \quad (17)$$

With the system of m linear equations with m unknowns $\vartheta(v_1), \vartheta(v_2), \dots, \vartheta(v_m)$, we can rearrange Eq (17) as follows:

$$\begin{aligned} \vartheta(v_1) &= 1 + \vartheta(v_1)[F(\eta - v_1 - Y_t) + w_1 f(\eta - Y_t)] + \sum_{j=2}^m w_j \vartheta(v_j) f(v_j + \eta - v_1 - Y_t) \\ \vartheta(v_2) &= 1 + \vartheta(v_1)[F(\eta - v_2 - Y_t) + w_1 f(v_1 + \eta - v_2 - Y_t)] + \sum_{j=2}^m w_j \vartheta(v_j) f(v_j + \eta - v_2 - Y_t) \\ &\vdots \\ \vartheta(v_m) &= 1 + \vartheta(v_1)[F(\eta - v_m - Y_t) + w_1 f(v_1 + \eta - v_m - Y_t)] + \sum_{j=2}^m w_j \vartheta(v_j) f(v_j + \eta - v_m - Y_t) \end{aligned}$$

This is equivalent to matrix form $\mathbf{J}_{m \times 1} = \mathbf{1}_{m \times 1} + \mathbf{R}_{m \times m} \mathbf{J}_{m \times 1}$. If the inverse $(\mathbf{I}_m - \mathbf{R}_{m \times m})^{-1}$ exists, then the unique solution is

$$\mathbf{J}_{m \times 1} = (\mathbf{I}_m - \mathbf{R}_{m \times m})^{-1} \mathbf{1}_{m \times 1},$$

where $\mathbf{J}_{m \times 1} = [\vartheta(v_1), \vartheta(v_2), \dots, \vartheta(v_m)]'$, $\mathbf{I}_m = \text{diag}(1, 1, \dots, 1)$ is the unit matrix order m , $\mathbf{1}_{m \times 1} = [1, 1, \dots, 1]'$ is a column vector of $\vartheta(v_j)$, and $\mathbf{R}_{m \times m}$ is a matrix of $(m, m)^{\text{th}}$ elements:

$$\mathbf{R}_{m \times m} = \begin{bmatrix} r_{11} & r_{12} & \dots & r_{1m} \\ r_{21} & r_{22} & \dots & r_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ r_{m1} & r_{m2} & \dots & r_{mm} \end{bmatrix}$$

where $r_{ij} = F(\eta - v_i - Y_t) + w_j f(v_j + \eta - v_i - Y_t)$; $i, j = 1, 2, \dots, m$.

Therefore, the approximation for the integral is obtained in the summation where v_j is replaced by φ in $\vartheta(v_j)$, which completes the proof.

Performance evaluation and comparison

A popular benchmark to measure the performance of a control chart is the ARL for detecting shifts in the process mean as a measure of its sensitivity and practicability. Thus, the performances of the analytical and numerical IE methods for determining the ARL for monitoring shifts in the process mean for a long-memory SARFIMAX(P, D, Q, r) process on a CUSUM control chart were compared. We determined the number of division points $m = 800$ for the approximated ARL by using the numerical IE method. The in-control situation comprises exponential white noise with mean $\beta_0 = 1$, while the out-of-control process is signaled by a shift in the process mean from β_0 to β_1 , where $\beta_1 = \beta_0(1 + \delta)$. Thus, $\delta = 0.01, 0.03, 0.05, 0.10, 0.30, 0.50, 0.70, 1.00, 1.50$, and 2.00 . $\delta = 0$, means that the process is in-control whereas $\delta > 0$, means that the process is out-of-control, denoted as ARL_0 and ARL_1 , respectively. Accordingly, ARL_0 was fixed at 370 or 500. For the CUSUM statistic, the value of η was determined as

Table 1. Calculated b for corresponding reference parameter η values for a long-memory SARFIMAX(P, D, Q, r)_s process on a CUSUM control chart for $ARL_0 = 370, 500$.

ARL ₀	Coefficients of models				Long-memory Models	η		
	Φ_1	Φ_2	Θ_1	ω_1		3.0	3.5	4.0
370	0.1	0.2	0.1	0.5	[1] SARFIMAX(2, 0.15, 1, 1) ₁₂	4.300562	3.498061	2.884593
	0.1	0.2	0.1	0.5	[2] SARFIMAX(2, 0.30, 1, 1) ₁₂	4.653230	3.704270	3.056922
	0.1		0.1	0.5	[3] SARFIMAX(1, 0.45, 1, 1) ₁₂	5.091801	3.888005	3.202990
	-0.1	0.2	0.1	0.5	[-1] SARFIMAX(2, 0.15, 1, 1) ₁₂	4.021805	3.304131	2.715860
	-0.1	0.2	0.1	0.5	[-2] SARFIMAX(2, 0.30, 1, 1) ₁₂	4.387763	3.553436	2.931650
	-0.1		0.1	0.5	[-3] SARFIMAX(1, 0.45, 1, 1) ₁₂	4.802330	3.776275	3.115083
500	0.1	0.2	0.1	0.5	[1] SARFIMAX(2, 0.15, 1, 1) ₁₂	4.677214	3.826880	3.199020
	0.1	0.2	0.1	0.5	[2] SARFIMAX(2, 0.30, 1, 1) ₁₂	5.080630	4.040850	3.374410
	0.1		0.1	0.5	[3] SARFIMAX(1, 0.45, 1, 1) ₁₂	5.698833	4.233519	3.523590
	-0.1	0.2	0.1	0.5	[-1] SARFIMAX(2, 0.15, 1, 1) ₁₂	4.375410	3.627221	3.027800
	-0.1	0.2	0.1	0.5	[-2] SARFIMAX(2, 0.30, 1, 1) ₁₂	4.774041	3.884130	3.246850
	-0.1		0.1	0.5	[-3] SARFIMAX(1, 0.45, 1, 1) ₁₂	5.266000	4.116095	3.433746

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3.0, 3.5, or 4.0. Thus, the CUSUM control limit (b) was calculated by using Eq (15) to achieve the specified ARL, the results for which are reported in Table 1. The coefficient parameters for long-memory SARFIMAX(2, 0.15, 1, 1)₁₂, SARFIMAX(2, 0.30, 1, 1)₁₂ and SARFIMAX(1, 0.45, 1, 1)₁₂ processes were set as $\Phi_1 = \pm 0.1$, $\Phi_2 = 0.2$, $\Theta_1 = 0.1$, and $\omega_1 = 0.5$. For a fixed ARL₀, the method achieving the smallest ARL₁ has delivered the best performance [1]. The results for the sensitivity, accuracy (see Definition 6), and computational times of the two methods are provided in Tables 2–6 and Fig 1.

Definition 6. The percentage accuracy (% Accuracy) indicates the relative performances of the analytical and numerical IE methods defined as follows:

$$\% \text{Accuracy} = 100 - \left| \frac{\ell(\varphi) - \vartheta(\varphi)}{\ell(\varphi)} \right| \times 100\%,$$

where $\ell(\varphi)$ and $\vartheta(\varphi)$ are the ARL values obtained using the analytical and numerical IE methods, respectively.

The following algorithm was developed by using the Mathematica program to calculate the values of the ARLs for a long-memory SARFIMAX(P, D, Q, r)_s process on a CUSUM control chart:

Step 1: Determine the values of coefficients Φ_1 , Φ_2 , Θ_1 , ω_1 , and the initial values of ξ_{t-s} , ξ_{t-2s} , $\dots$, ξ_{t-Qs} , X_{lt} , Y_{t-s} , Y_{t-2s} , Y_{t-3s} , $\dots$, Y_{t-Ps} , $Y_{t-(P+1)s}$.

Step 2: Specify β_0 for the in-control exponential white noise ($\xi_t \sim \text{Exp}(\beta)$).

Step 3: Determine the value of known parameter (η) and the initial value of CUSUM statistic φ .

Step 4: Calculate the upper control limit (b) by using Eq (15) for the specified ARL₀ and η .

Step 5: Compute ARL₁ for shifts in the process mean (β_1) where b is attained from Step 4.

The above algorithm was executed to obtain ARLs for out-of-control situations as measurements of the performance of the proposed analytical and numerical IE methods.

The results for computed parameter b and reference parameter η for ARL₀ = 370 or 500 using the proposed ARL schemes are reported in Table 1. b is inversely proportional to η for

Table 2. Comparison of the ARL_1 values obtained using the analytical and numerical ARL methods for long-memory SARFIMAX(P, D, Q, r) processes with exponential white noise on a CUSUM control chart for $ARL_0 = 370$ with $\Phi_1 = 0.1$, $\Phi_2 = 0.2$, $\Theta_1 = 0.1$, and $\omega_1 = 0.5$.

δ	η	3.0			3.5			4.0		
	b	4.300562	4.65323	5.091801	3.498061	3.70427	3.888005	2.884593	3.056922	3.20299
	Model	[1]	[2]	[3]	[1]	[2]	[3]	[1]	[2]	[3]
0.01	$\ell(\varphi)$	345.218	343.883	341.545	347.171	346.781	346.375	348.003	347.812	347.626
	(Sec.)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)
	$\vartheta(\varphi)$	344.460	343.135	340.870	346.481	346.065	345.639	347.410	347.190	346.979
	(Min.)	(32.22)	(32.11)	(31.81)	(31.99)	(32.62)	(31.65)	(31.89)	(32.76)	(31.70)
	%Accuracy	99.78	99.78	99.80	99.80	99.79	99.79	99.83	99.82	99.81
0.03	$\ell(\varphi)$	301.755	298.353	292.454	306.773	305.765	304.718	308.935	308.438	307.954
	(Sec.)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)
	$\vartheta(\varphi)$	301.116	297.734	291.919	306.179	305.151	304.090	308.421	307.899	307.395
	(Min.)	(31.99)	(31.93)	(31.77)	(32.09)	(32.64)	(31.64)	(32.11)	(32.82)	(31.76)
	%Accuracy	99.79	99.79	99.82	99.81	99.80	99.79	99.83	99.83	99.82
0.05	$\ell(\varphi)$	265.144	260.308	251.990	272.343	270.888	269.383	275.478	274.756	274.053
	(Sec.)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)
	$\vartheta(\varphi)$	264.602	259.793	251.564	271.830	270.359	268.844	275.031	274.287	273.567
	(Min.)	(31.72)	(32.44)	(31.76)	(31.75)	(32.31)	(31.46)	(31.73)	(32.02)	(31.66)
	%Accuracy	99.80	99.80	99.83	99.81	99.80	99.80	99.84	99.83	99.82
0.10	$\ell(\varphi)$	195.987	189.360	178.168	206.079	204.013	201.891	210.588	209.541	208.526
	(Sec.)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)
	$\vartheta(\varphi)$	195.621	189.027	177.924	205.714	203.640	201.515	210.265	209.203	208.178
	(Min.)	(31.64)	(31.91)	(31.65)	(31.55)	(33.22)	(31.56)	(31.49)	(31.86)	(31.48)
	%Accuracy	99.81	99.82	99.86	99.82	99.82	99.81	99.85	99.84	99.83
0.30	$\ell(\varphi)$	74.675	69.144	60.371	83.787	81.838	79.886	88.238	87.176	86.165
	(Sec.)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)
	$\vartheta(\varphi)$	74.576	69.068	60.343	83.671	81.722	79.772	88.129	87.063	86.050
	(Min.)	(31.70)	(31.78)	(31.45)	(31.59)	(31.99)	(31.74)	(31.71)	(31.90)	(31.81)
	%Accuracy	99.87	99.89	99.95	99.86	99.86	99.86	99.88	99.87	99.87
0.50	$\ell(\varphi)$	37.626	34.278	29.236	43.515	42.211	40.930	46.607	45.853	45.146
	(Sec.)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)
	$\vartheta(\varphi)$	37.590	34.254	29.236	43.467	42.164	40.884	46.559	45.804	45.096
	(Min.)	(31.90)	(31.90)	(31.77)	(31.45)	(32.97)	(31.48)	(31.73)	(32.70)	(31.51)
	%Accuracy	99.90	99.93	100.00	99.89	99.89	99.89	99.90	99.89	99.89
0.70	$\ell(\varphi)$	22.750	20.717	17.792	26.538	25.674	24.839	28.651	28.126	27.640
	(Sec.)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)
	$\vartheta(\varphi)$	22.734	20.708	17.796	26.514	25.650	24.818	28.626	28.102	27.615
	(Min.)	(32.23)	(32.27)	(31.69)	(31.65)	(32.99)	(31.54)	(31.78)	(33.22)	(31.64)
	%Accuracy	99.93	99.96	99.98	99.91	99.91	99.92	99.91	99.91	99.91
1.00	$\ell(\varphi)$	13.295	12.259	10.858	15.382	14.888	14.421	16.641	16.322	16.030
	(Sec.)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)
	$\vartheta(\varphi)$	13.289	12.255	10.861	15.372	14.877	14.411	16.630	16.310	16.018
	(Min.)	(31.56)	(31.71)	(31.77)	(31.50)	(31.49)	(31.53)	(31.49)	(31.61)	(31.65)
	%Accuracy	99.95	99.97	99.97	99.93	99.93	99.93	99.93	99.93	99.93
1.50	$\ell(\varphi)$	7.543	7.140	6.651	8.460	8.230	8.020	9.077	8.916	8.771
	(Sec.)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)
	$\vartheta(\varphi)$	7.541	7.139	6.652	8.456	8.227	8.017	9.072	8.911	8.767
	(Min.)	(31.97)	(31.53)	(31.52)	(31.47)	(31.49)	(31.72)	(31.72)	(31.69)	(31.82)

(Continued)

Table 2. (Continued)

δ	η	3.0			3.5			4.0		
	b	4.300562	4.65323	5.091801	3.498061	3.70427	3.888005	2.884593	3.056922	3.20299
	Model	[1]	[2]	[3]	[1]	[2]	[3]	[1]	[2]	[3]
2.0	%Accuracy	99.97	99.99	99.98	99.95	99.96	99.96	99.94	99.94	99.95
	$\ell(\varphi)$	5.296	5.115	4.924	5.765	5.641	5.531	6.112	6.020	5.937
	(Sec.)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)
	$\vartheta(\varphi)$	5.295	5.114	4.924	5.763	5.639	5.529	6.110	6.017	5.935
	(Min.)	(31.58)	(31.90)	(32.02)	(31.51)	(31.63)	(31.89)	(31.59)	(31.59)	(31.93)
	%Accuracy	99.78	99.78	99.80	99.80	99.79	99.79	99.83	99.82	99.81

The results are expressed as percentage accuracy with computational times in parentheses for the analytical IE method (seconds) and the numerical IE method (minutes).

[1] SARFIMAX(2, 0.15, 1, 1)₁₂, [2] SARFIMAX(2, 0.30, 1, 1)₁₂ and [3] SARFIMAX(1, 0.45, 1, 1)₁₂

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Table 3. Comparison of the ARL₁ values obtained using the analytical and numerical ARL methods for long-memory SARFIMAX(P, D, Q, r)_s processes with exponential white noise on a CUSUM control chart for ARL₀ = 370 with $\Phi_1 = -0.1$, $\Phi_2 = 0.2$, $\Theta_1 = 0.1$, and $\omega_1 = 0.5$.

δ	η	3.0			3.5			4.0		
	b	4.021805	4.387763	4.80233	3.304131	3.553436	3.776275	2.71586	2.93165	3.115083
	Model	[-1]	[-2]	[-3]	[-1]	[-2]	[-3]	[-1]	[-2]	[-3]
0.01	$\ell(\varphi)$	346.040	344.922	343.186	347.482	347.072	346.628	348.162	347.953	347.741
	(Sec.)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)
	$\vartheta(\varphi)$	345.294	344.164	342.454	346.820	346.375	345.904	347.599	347.352	347.109
	(Min.)	(31.80)	(31.56)	(31.88)	(32.35)	(31.53)	(31.91)	(31.91)	(31.61)	(32.01)
	%Accuracy	99.78	99.78	99.79	99.81	99.80	99.79	99.84	99.83	99.82
0.03	$\ell(\varphi)$	303.857	301.000	296.592	307.581	306.518	305.372	309.351	308.807	308.253
	(Sec.)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)
	$\vartheta(\varphi)$	303.223	300.363	295.993	307.010	305.918	304.752	308.863	308.286	307.706
	(Min.)	(31.95)	(31.79)	(31.73)	(32.08)	(31.57)	(31.45)	(31.70)	(31.91)	(31.56)
	%Accuracy	99.79	99.79	99.80	99.81	99.80	99.80	99.84	99.83	99.82
0.05	$\ell(\varphi)$	268.149	264.067	257.818	273.512	271.975	270.323	276.085	275.292	274.486
	(Sec.)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)
	$\vartheta(\varphi)$	267.606	263.529	257.325	273.016	271.457	269.790	275.659	274.838	274.011
	(Min.)	(31.69)	(31.83)	(31.67)	(31.93)	(31.67)	(31.60)	(32.13)	(31.79)	(31.74)
	%Accuracy	99.80	99.80	99.81	99.82	99.81	99.80	99.85	99.84	99.83
0.10	$\ell(\varphi)$	200.161	194.503	185.987	207.750	205.554	203.214	211.473	210.317	209.151
	(Sec.)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)
	$\vartheta(\varphi)$	199.786	194.142	185.676	207.396	205.187	202.840	211.164	209.990	208.809
	(Min.)	(32.07)	(31.65)	(31.57)	(31.93)	(31.47)	(31.43)	(31.68)	(31.52)	(31.45)
	%Accuracy	99.81	99.81	99.83	99.83	99.82	99.82	99.85	99.84	99.84
0.30	$\ell(\varphi)$	78.327	73.410	66.434	85.402	83.288	81.098	89.151	87.962	86.786
	(Sec.)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)
	$\vartheta(\varphi)$	78.217	73.315	66.371	85.286	83.172	80.983	89.045	87.851	86.672
	(Min.)	(31.79)	(31.52)	(31.62)	(31.65)	(31.40)	(31.61)	(32.02)	(31.58)	(31.52)
	%Accuracy	99.86	99.87	99.91	99.86	99.86	99.86	99.88	99.87	99.87
0.50	$\ell(\varphi)$	39.925	36.847	32.690	44.618	43.179	41.722	47.263	46.410	45.579
	(Sec.)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)
	$\vartheta(\varphi)$	39.882	36.812	32.672	44.569	43.130	41.675	47.217	46.361	45.530
	(Min.)	(32.24)	(31.93)	(31.69)	(32.34)	(31.56)	(31.60)	(32.75)	(31.58)	(31.55)

(Continued)

Table 3. (Continued)

δ	η	3.0			3.5			4.0		
	b	4.021805	4.387763	4.80233	3.304131	3.553436	3.776275	2.71586	2.93165	3.115083
	Model	[-1]	[-2]	[-3]	[-1]	[-2]	[-3]	[-1]	[-2]	[-3]
0.70	%Accuracy	99.89	99.91	99.94	99.89	99.89	99.89	99.90	99.89	99.89
	$\ell(\varphi)$	24.195	22.270	19.779	27.280	26.313	25.354	29.113	28.513	27.937
	(Sec.)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)
	$\vartheta(\varphi)$	24.174	22.255	19.774	27.255	26.289	25.331	29.089	28.488	27.913
	(Min.)	(31.93)	(31.75)	(31.71)	(33.06)	(31.70)	(31.73)	(33.29)	(31.74)	(31.61)
1.00	%Accuracy	99.91	99.93	99.97	99.91	99.91	99.91	99.92	99.91	99.91
	$\ell(\varphi)$	14.067	13.045	11.799	15.816	15.253	14.707	16.926	16.556	16.208
	(Sec.)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)
	$\vartheta(\varphi)$	14.058	13.039	11.797	15.805	15.242	14.697	16.915	16.545	16.196
	(Min.)	(31.93)	(32.00)	(31.83)	(32.00)	(31.76)	(31.52)	(32.03)	(32.02)	(31.58)
1.50	%Accuracy	99.94	99.95	99.98	99.93	99.93	99.93	99.94	99.93	99.93
	$\ell(\varphi)$	7.866	7.442	6.972	8.667	8.399	8.149	9.223	9.034	8.859
	(Sec.)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)
	$\vartheta(\varphi)$	7.863	7.440	6.972	8.663	8.395	8.145	9.219	9.029	8.854
	(Min.)	(31.82)	(31.80)	(31.60)	(32.07)	(31.82)	(31.58)	(32.05)	(31.86)	(31.74)
2.0	%Accuracy	99.96	99.97	100.00	99.95	99.95	99.95	99.96	99.94	99.94
	$\ell(\varphi)$	5.453	5.249	5.046	5.879	5.731	5.598	6.198	6.088	5.987
	(Sec.)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)
	$\vartheta(\varphi)$	5.451	5.248	5.046	5.877	5.730	5.596	6.196	6.085	5.985
	(Min.)	(31.67)	(31.69)	(31.73)	(32.97)	(31.59)	(31.45)	(33.21)	(31.57)	(31.61)
%Accuracy		99.78	99.78	99.79	99.81	99.80	99.79	99.84	99.83	99.82

The results are expressed as percentage accuracy with computational times in parentheses for the analytical IE method (seconds) and the numerical IE method (minutes).

[-1] SARFIMAX(2, 0.15, 1, 1)₁₂, [-2] SARFIMAX(2, 0.30, 1, 1)₁₂ and [-3] SARFIMAX(1, 0.45, 1, 1)₁₂

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Table 4. Comparison of the ARL_1 values obtained using the analytical and numerical ARL methods for long-memory SARFIMAX(P, D, Q, r)_s processes with exponential white noise on a CUSUM control chart for $ARL_0 = 500$ with $\Phi_1 = 0.1$, $\Phi_2 = 0.2$, $\Theta_1 = 0.1$, and $\omega_1 = 0.5$.

δ	η	3.0			3.5			4.0		
	b	4.677214	5.08063	5.698833	3.82688	4.04085	4.233519	3.19902	3.37441	3.52359
	Model	[1]	[2]	[3]	[1]	[2]	[3]	[1]	[2]	[3]
0.01	$\ell(\varphi)$	464.199	461.718	455.617	467.408	466.785	466.133	468.690	468.395	468.107
	(Sec.)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)
	$\vartheta(\varphi)$	463.088	460.632	454.740	466.388	465.729	465.052	467.803	467.468	467.147
	(Min.)	(33.10)	(32.63)	(32.83)	(33.07)	(32.62)	(33.01)	(33.31)	(32.94)	(33.00)
	%Accuracy	99.76	99.76	99.81	99.78	99.77	99.77	99.81	99.80	99.79
0.03	$\ell(\varphi)$	401.849	395.580	380.336	410.025	408.439	406.773	413.348	412.586	411.842
	(Sec.)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)
	$\vartheta(\varphi)$	400.926	394.702	379.704	409.155	407.542	405.859	412.584	411.789	411.019
	(Min.)	(33.14)	(32.73)	(33.09)	(32.87)	(32.47)	(32.62)	(33.33)	(33.00)	(33.01)
	%Accuracy	99.77	99.78	99.83	99.79	99.78	99.78	99.82	99.81	99.80

(Continued)

Table 4. (Continued)

δ	η	3.0			3.5			4.0		
	b	4.677214	5.08063	5.698833	3.82688	4.04085	4.233519	3.19902	3.37441	3.52359
	Model	[1]	[2]	[3]	[1]	[2]	[3]	[1]	[2]	[3]
0.05	$\ell(\varphi)$	349.815	340.971	319.685	361.464	359.193	356.813	366.257	365.156	364.082
	(Sec.)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)
	$\vartheta(\varphi)$	349.043	340.257	319.234	360.717	358.427	356.035	365.596	364.467	363.372
	(Min.)	(34.88)	(33.10)	(33.10)	(33.48)	(33.72)	(32.70)	(33.54)	(32.68)	(32.60)
	%Accuracy	99.78	99.79	99.86	99.79	99.79	99.78	99.82	99.81	99.80
0.10	$\ell(\varphi)$	252.977	241.068	213.082	269.050	265.877	262.570	275.852	274.279	272.752
	(Sec.)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)
	$\vartheta(\varphi)$	252.471	240.631	212.901	268.529	265.348	262.040	275.382	273.791	272.251
	(Min.)	(33.53)	(32.76)	(32.82)	(33.34)	(32.73)	(32.77)	(33.60)	(32.84)	(33.03)
	%Accuracy	99.80	99.82	99.92	99.81	99.80	99.80	99.83	99.82	99.82
0.30	$\ell(\varphi)$	89.915	80.523	60.164	103.720	100.865	97.964	110.139	108.614	107.159
	(Sec.)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)
	$\vartheta(\varphi)$	89.792	80.444	60.213	103.563	100.711	97.816	109.988	108.459	107.003
	(Min.)	(34.90)	(32.97)	(32.91)	(33.57)	(32.55)	(32.76)	(33.91)	(32.83)	(32.98)
	%Accuracy	99.86	99.90	99.92	99.85	99.85	99.85	99.86	99.86	99.85
0.50	$\ell(\varphi)$	43.245	37.792	26.715	51.847	50.000	48.162	56.161	55.113	54.128
	(Sec.)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)
	$\vartheta(\varphi)$	43.204	37.774	26.757	51.784	49.94	48.107	56.097	55.049	54.064
	(Min.)	(34.82)	(32.66)	(33.08)	(34.70)	(32.70)	(33.02)	(35.00)	(32.58)	(32.95)
	%Accuracy	99.91	99.95	99.84	99.88	99.88	99.89	99.89	99.88	99.88
0.70	$\ell(\varphi)$	25.368	22.161	15.994	30.748	29.556	28.391	33.623	32.912	32.251
	(Sec.)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)
	$\vartheta(\varphi)$	25.352	22.156	16.025	30.718	29.528	28.365	33.591	32.88	32.220
	(Min.)	(32.81)	(32.77)	(32.98)	(32.48)	(32.64)	(33.01)	(32.47)	(32.78)	(33.28)
	%Accuracy	99.94	99.98	99.81	99.90	99.91	99.91	99.90	99.90	99.90
1.00	$\ell(\varphi)$	14.439	12.861	10.048	17.311	19.649	16.016	18.976	18.555	18.169
	(Sec.)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)
	$\vartheta(\varphi)$	14.433	12.853	10.041	17.299	16.637	16.006	18.962	18.541	18.156
	(Min.)	(32.47)	(32.80)	(33.11)	(32.51)	(32.78)	(33.02)	(32.56)	(32.92)	(33.25)
	%Accuracy	99.96	99.94	99.93	99.93	84.67	99.94	99.93	99.92	99.93
1.50	$\ell(\varphi)$	8.033	7.444	6.515	9.250	8.953	8.678	10.040	9.835	9.650
	(Sec.)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)
	$\vartheta(\varphi)$	8.031	7.435	6.520	9.246	8.949	8.675	10.035	9.83	9.645
	(Min.)	(37.45)	(33.08)	(32.76)	(37.73)	(32.96)	(32.46)	(37.45)	(32.91)	(32.74)
	%Accuracy	99.98	99.88	99.92	99.96	99.96	99.97	99.95	99.95	99.95
2.0	$\ell(\varphi)$	5.593	5.337	4.989	6.201	6.045	5.904	6.637	6.521	6.418
	(Sec.)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)
	$\vartheta(\varphi)$	5.592	5.337	4.991	6.199	6.043	5.903	6.635	6.518	6.416
	(Min.)	(36.97)	(32.85)	(32.92)	(37.36)	(32.87)	(32.62)	(37.61)	(32.86)	(32.90)
	%Accuracy	99.76	99.76	99.81	99.78	99.77	99.77	99.81	99.80	99.79

The results are expressed as percentage accuracy with computational times in parentheses for the analytical IE method (seconds) and the numerical IE method (minutes).

[1] SARFIMAX(2, 0.15, 1, 1)₁₂, [2] SARFIMAX(2, 0.30, 1, 1)₁₂ and [3] SARFIMAX(1, 0.45, 1, 1)₁₂

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Table 5. Comparison of the ARL_1 values obtained using the analytical and numerical ARL methods for long-memory SARFIMAX(P, D, Q, r) processes with exponential white noise on a CUSUM control chart for $ARL_0 = 500$ with $\Phi_1 = -0.1$, $\Phi_2 = 0.2$, $\Theta_1 = 0.1$, and $\omega_1 = 0.5$.

δ	η	3.0			3.5			4.0		
	b	4.375410	4.774041	5.266000	3.627221	3.884130	4.116095	3.027800	3.246850	3.433746
	Model	[-1]	[-2]	[-3]	[-1]	[-2]	[-3]	[-1]	[-2]	[-3]
0.01	$\ell(\varphi)$	465.589	463.677	460.320	467.884	467.245	466.541	468.932	468.612	468.284
	(Sec.)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)
	$\vartheta(\varphi)$	464.493	462.566	459.271	466.903	466.215	465.474	468.086	467.714	467.344
	(Min.)	(33.38)	(33.31)	(34.02)	(33.45)	(33.11)	(33.96)	(33.70)	(33.20)	(33.98)
%Accuracy		99.76	99.76	99.77	99.79	99.78	99.77	99.82	99.81	99.80
0.03	$\ell(\varphi)$	405.380	400.527	391.952	411.268	409.621	407.816	413.982	413.149	412.301
	(Sec.)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)
	$\vartheta(\varphi)$	404.458	399.609	391.121	410.428	408.743	406.912	413.252	412.377	411.493
	(Min.)	(33.31)	(33.20)	(34.00)	(32.65)	(32.92)	(33.65)	(33.23)	(33.39)	(34.05)
%Accuracy		99.77	99.77	99.79	99.80	99.79	99.78	99.82	99.81	99.80
0.05	$\ell(\varphi)$	354.826	347.946	335.848	363.256	360.888	358.304	367.177	365.971	364.744
	(Sec.)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)
	$\vartheta(\varphi)$	354.046	347.181	335.186	362.532	360.135	357.532	366.544	365.302	364.047
	(Min.)	(32.95)	(32.58)	(33.43)	(32.61)	(32.50)	(34.70)	(32.90)	(32.48)	(33.56)
%Accuracy		99.78	99.78	99.80	99.80	99.79	99.78	99.83	99.82	99.81
0.10	$\ell(\varphi)$	259.828	250.440	234.225	271.581	268.246	264.639	277.176	275.444	273.693
	(Sec.)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)
	$\vartheta(\varphi)$	259.301	249.946	233.842	271.072	267.722	264.109	276.725	274.969	273.199
	(Min.)	(32.77)	(32.62)	(33.38)	(32.90)	(32.51)	(33.56)	(32.95)	(32.58)	(33.59)
%Accuracy		99.80	99.80	99.84	99.81	99.80	99.80	99.84	99.83	99.82
0.30	$\ell(\varphi)$	95.610	87.865	75.343	106.059	102.991	99.771	111.449	109.742	108.054
	(Sec.)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)
	$\vartheta(\varphi)$	95.468	87.751	75.293	105.902	102.835	99.619	111.301	109.590	107.898
	(Min.)	(34.60)	(32.70)	(33.70)	(34.04)	(32.38)	(33.44)	(34.33)	(32.77)	(33.56)
%Accuracy		99.85	99.87	99.93	99.85	99.85	99.85	99.87	99.86	99.86
0.50	$\ell(\varphi)$	46.698	42.031	34.889	53.391	51.372	49.302	57.073	55.887	54.732
	(Sec.)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)
	$\vartheta(\varphi)$	46.646	41.995	34.855	53.327	51.310	49.244	57.011	55.826	54.668
	(Min.)	(34.46)	(35.26)	(34.11)	(34.03)	(32.48)	(33.65)	(34.29)	(32.74)	(33.73)
%Accuracy		99.89	99.91	99.90	99.88	99.88	99.88	99.89	99.89	99.88
0.70	$\ell(\varphi)$	27.477	24.642	20.506	31.762	30.439	29.111	34.249	33.436	32.655
	(Sec.)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)
	$\vartheta(\varphi)$	27.453	24.628	20.509	31.730	30.409	29.084	34.218	33.406	32.623
	(Min.)	(34.20)	(32.67)	(33.34)	(34.15)	(32.47)	(33.45)	(34.21)	(32.90)	(33.45)
%Accuracy		99.91	99.94	99.99	99.90	99.90	99.91	99.91	99.91	99.90
1.00	$\ell(\varphi)$	15.529	14.073	12.081	17.887	17.138	16.406	19.352	18.864	18.405
	(Sec.)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)
	$\vartheta(\varphi)$	15.520	14.069	12.083	17.874	17.126	16.395	19.338	18.850	18.391
	(Min.)	(33.38)	(33.10)	(33.61)	(32.76)	(32.80)	(33.72)	(33.83)	(32.84)	(34.34)
%Accuracy		99.94	99.97	99.98	99.93	99.93	99.93	99.93	99.93	99.92
1.50	$\ell(\varphi)$	8.473	7.891	7.173	9.517	9.172	8.846	10.228	9.985	9.762
	(Sec.)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)
	$\vartheta(\varphi)$	8.470	7.890	7.174	9.512	9.168	8.843	10.223	9.980	9.757
	(Min.)	(33.52)	(34.36)	(33.16)	(32.95)	(33.34)	(32.61)	(33.02)	(33.67)	(32.94)

(Continued)

Table 5. (Continued)

δ	η	3.0			3.5			4.0		
	b	4.375410	4.774041	5.266000	3.627221	3.884130	4.116095	3.027800	3.246850	3.433746
	Model	[-1]	[-2]	[-3]	[-1]	[-2]	[-3]	[-1]	[-2]	[-3]
2.0	%Accuracy	99.96	99.99	99.99	99.95	99.96	99.97	99.95	99.95	99.95
	$\ell(\varphi)$	5.802	5.529	5.228	6.345	6.159	5.990	6.745	6.606	6.480
	(Sec.)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)
	$\vartheta(\varphi)$	5.801	5.528	5.229	6.343	6.157	5.988	6.742	6.604	6.478
	(Min.)	(32.79)	(34.17)	(32.53)	(32.77)	(33.69)	(32.46)	(32.80)	(33.71)	(32.69)
	%Accuracy	99.76	99.76	99.77	99.79	99.78	99.77	99.82	99.81	99.80

The results are expressed as percentage accuracy with computational times in parentheses for the analytical IE method (seconds) and the numerical IE method (minutes).

[-1] SARFIMAX(2, 0.15, 1, 1)₁₂, [-2] SARFIMAX(2, 0.30, 1, 1)₁₂ and [-3] SARFIMAX(1, 0.45, 1, 1)₁₂

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Table 6. The SARFIMAX(P, D, Q, r)₁₂ coefficients for the PTT stock price dataset from January 2003 to August 2018.

Variable	Coefficient	Std. Error	t-Statistic	P-value
D(PTT, 0, 12)	0.496151	0.017080	29.04865	0.0000
EUR	-0.114685	0.054130	-2.118690	0.0355
SAR(12)	1.000000	0.035469	28.19334	0.0000
SMA(12)	0.799986	0.000298	2682.302	0.0000

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long-memory process models [1] to [3]. For instance, for prespecified $ARL_0 = 370$ on long-memory SARFIMAX(2, 0.15, 1, 1)₁₂ with coefficient values $\Phi_1 = \pm 0.1$, $\Phi_2 = 0.2$, $\Theta_1 = 0.1$, and $\omega_1 = 0.5$, it can be seen that for $\eta = 3$, $b = 4.300562$ and 4.021805 , respectively. Only the ARL_0 , b , and η values are involved in the design of the CUSUM control chart (see Eq (15)).

The performance assessment of the analytical and numerical IE methods for determining the ARL for detecting shifts in the process means requires computing their ARL_1 values, as described earlier. Tables 2–5 provide the numerical results obtain from both methods in which we report their ARL_1 values, percentage accuracy, and computational times. We summarize our findings as follows:

1. The results of the analytical and numerical IE methods for detecting changes in the process mean tended to decrease when the shift size was increased in the order of long-memory process models [1] SARFIMAX(2, 0.15, 1, 1)₁₂, [2] SARFIMAX(2, 0.30, 1, 1)₁₂ and [3] SARFIMAX(1, 0.45, 1, 1)₁₂.
2. The ARL_1 values for [3] SARFIMAX(1, 0.45, 1, 1)₁₂ and [2] SARFIMAX(2, 0.30, 1, 1)₁₂ were slightly lower than for [1] SARFIMAX(2, 0.15, 1, 1)₁₂. With negative coefficients, the results were similar: [-3] SARFIMAX(1, 0.45, 1, 1)₁₂ and [-2] SARFIMAX(2, 0.30, 1, 1)₁₂ were slightly lower than those [-1] SARFIMAX(2, 0.15, 1, 1)₁₂ (see the results for $ARL_0 = 370$ in Tables 2 and 3 and 500 in Tables 4 and 5).
3. The reference value (η) was inversely proportional to the upper control limit and directly proportional to ARL_1 for both methods.
4. The percentage accuracy results were ~99% in all cases, meaning that the proposed analytical IE method was very accurate.

Fig 1. Comparative performances of the analytical and numerical IEs for determining ARL_1 for detecting shifts in the mean of a long-memory process on a CUSUM control chart given reference value (η) = 3.0, 3.5 and 4.0. (A) Models [1] to [3] for $ARL_0 = 370$, (B) models [-1] to [-3] for $ARL_0 = 370$, (C) models [1] to [3] for $ARL_0 = 500$, and (D) models [-1] to [-3] for $ARL_0 = 500$.

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- The values in parentheses in each column are the computational times for calculating ARL_1 : 31–34 minutes for $ARL_0 = 370$ and 32–38 minutes for $ARL_0 = 500$ using the numerical IE but less than 1 second for both using the analytical IE.

Fig 1 shows plots of δ (shift size) versus ARL_1 value obtained by using the analytical IE for the various long-memory SARFIMAX(P, D, Q, r)_s processes with different reference values (η) = 3.0, 3.5, 4.0 and for $ARL_0 = 370$ and 500. These graphs demonstrate the various characteristics of long-memory processes on the CUSUM control chart with coefficients $\Phi_1 = 0.1$, $\Phi_2 = 0.2$, $\Theta_1 = 0.1$, and $\omega_1 = 0.5$ in Fig 1A and 1C and $\Phi_1 = -0.1$, $\Phi_2 = 0.2$, $\Theta_1 = 0.1$, and $\omega_1 = 0.5$ in Fig 1B and 1D. It is evident that ARL_1 decreased rapidly for small shifts ($0 < \delta \leq 0.05$) and slightly less so for moderate shifts ($0.05 < \delta \leq 0.20$). In addition, the lowest reference parameter (η) = 3.0 was better for detecting small to moderate shifts in the process mean than 3.5 and 4. Moreover, when D was increased from 0.15 to 0.30, detecting changes was more sensitive. The results also show that the efficacy of both IEs for detecting changes was best for models [3] and [-3] with reference parameter (η) = 3.0 and = 370 and 500.

In summary, the results show that the analytical IE method is a good alternative to the numerical IE method for evaluating shifts in the process mean of a long-memory SARFIMAX(P, D, Q, r)_s process with exponential white noise on a CUSUM control chart, especially because of the marked reduction in computational time.

Fig 2. The fitted SARFIMAX(1, 0.496151, 1 1)₁₂ model.

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Illustrative example

In this section, the results of applying the proposed analytical and numerical IE methods for determining the ARL for monitoring changes in the mean of a long-memory SARFIMAX(P, D, Q, r)_s processes with exponential white noise on a CUSUM control chart involving real data are provided. The real dataset comprises monthly stock price data for the PTT Public Company Ltd. in Thailand (<https://th.investing.com/equities/ptt-historical-data>) with the EUR/THB exchange rate as the exogenous variable (<https://th.investing.com/currencies/eur-thb-historical-data>). In this example, we used monthly data from January 2003 to August 2018 comprising 188 observations. A seasonal pattern exists when a series is influenced by a seasonal factor, which is a 12-month difference term in the model ($S = 12$). Accordingly, SARFIMAX(P, D, Q, r)₁₂ was selected as the model. Statistical software package Eviews 10 was utilized for SARFIMAX(P, D, Q, r)₁₂ filtering and estimating the model parameters.

The results in Table 6 reveal that the probability of variable X was significant (0.05; p -value = 0.0355), which indicates that the exchange rate does impact the stock price data for PTT. It can also be seen that the dataset is suitable for a long-memory SARFIMAX(1, 0.496151, 1 1)₁₂ model presenting statistically significant parameters with coefficients $\Phi_1 = 1.00$, $\Theta_1 = 0.799986$, $\omega_1 = -0.114685$ and the residual series behaving as white noise. The suitability of the model was checked by plotting the actual (Y_t) and fitted ($\hat{Y}_t$) components and appears to be in close agreement, as shown in Fig 2.

After that, the distribution of the white noise was confirmed as asymptotic exponential by using the Kolmogorov-Smirnov test. The results show that the white noise significantly fitted an exponential distribution (P -value = 0.091 > 0.05) with a mean of 3.42526 when the process was in-control, as reported in Table 7.

Table 7. Testing the suitability of an exponential distribution for the exponential white noise.

Testing whether the white noise is exponentially distributed. Alternative hypothesis: two-sided	
One-sample Kolmogorov-Smirnov test	1.24200
Exponential parameter	3.42526
Asymp. Sig. (2-tailed)	0.09100*

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Table 8. Comparison of the ARL_1 values obtained via the analytical and numerical IE methods for detecting changes in the mean of the long-memory SARFIMAX(1, 0.496151, 1 1)₁₂ process on a CUSUM control chart for $\Phi_1 = 1.00$, $\Theta_1 = 0.799986$, and $\omega_1 = -0.114685$.

ARL_0	ARL	δ									
		0.01	0.03	0.05	0.10	0.3	0.5	0.7	1.0	1.5	2.0
370	$\ell(\varphi)$	361.218	344.416	328.571	292.746	190.263	129.346	91.510	58.135	31.553	19.744
	(Sec.)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)
	$\vartheta(\varphi)$	360.476	343.724	327.926	292.203	189.987	129.205	91.439	58.113	31.557	19.753
	(Min.)	(33.44)	(33.51)	(33.36)	(33.49)	(32.86)	(32.57)	(32.42)	(34.02)	(32.82)	(32.72)
	%Accuracy	99.79	99.80	99.80	99.81	99.85	99.89	99.92	99.96	99.99	99.95
500	$\ell(\varphi)$	486.626	461.144	437.247	383.728	235.376	151.703	102.263	61.059	30.789	18.538
	(Sec.)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)
	$\vartheta(\varphi)$	485.592	460.198	436.382	383.037	235.104	151.615	102.256	61.095	30.833	18.573
	(Min.)	(33.07)	(33.01)	(32.45)	(34.14)	(32.90)	(33.01)	(32.65)	(32.89)	(32.94)	(32.77)
	%Accuracy	99.79	99.79	99.80	99.82	99.88	99.94	99.99	99.94	99.86	99.81

The results are expressed as percentage accuracy with the computational times in parentheses for the analytical IE (seconds) and numerical IE (minutes) methods.

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Based on the results in Tables 6 and 7, the SARFIMAX(1, 0.496151, 1 1)₁₂ model (Y_t) comprised the following equation:

$$Y_t = Y_{t-12} - 0.496151Y_{t-24} - 0.124999Y_{t-36} - 0.06266Y_{t-48} + 0.49615Y_{t-12} + 0.12499Y_{t-24} + 0.06266Y_{t-36} + \xi_t - 0.799986\xi_{t-12} - 0.114685X_t,$$

where $\xi_t \sim \text{Exp}(\beta_0 = 3.42526)$.

Next, we calculated ARL_0 using the analytical and numerical IEs and chose the value of parameter (η) = 6.5 after calculating the control limits for $ARL_0 = 370$ and 500 using Eq (16) ($b = 17.37735$ and 19.2241, respectively). The rationale behind the use of 370 and 500 of the in-control ARLs is that they give good results in practice. The performances of the two methods for determining the ARL for detecting shifts in the process mean of a long-memory SARFIMAX(1, 0.496151, 1 1)₁₂ on a CUSUM control chart are presented in Table 8 and Fig 3.

It can be seen that the ARL_1 results using the two methods are similar to those in Tables 2–5. The numerical results obtained from both methods were similar for both short- and long-term detections ($ARL_0 = 370$ and 500, respectively) for all shift sizes of the process mean. Once again, the computation time for the analytical IE was far less than that of the numerical IE

Fig 3. Detecting shifts the mean on the long-memory SARFIMAX(1, 0.496151, 1 1)₁₂ process based on real data on an upper-side CUSUM control chart for (A) $ARL_0 = 370$ and (B) $ARL_0 = 500$. CL, center line; UCL, upper control limit; LCL, lower control limit.

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Table 9. Comparison of the ARL_1 values obtained via the analytical IE on a CUSUM control chart and a numerical IE method on a EWMA control chart with $\lambda = 0.1$ for detecting changes in the mean of the long-memory SARFIMAX(1, 0.496151, 1 1)₁₂ process.

ARL_0	b	Control charts	δ									
			0.01	0.03	0.05	0.10	0.3	0.5	0.7	1.0	1.5	2.0
370	17.37735	CUSUM	361.218	344.416	328.571	292.746	190.263	129.346	91.510	58.135	31.553	19.744
	0.2507646	EWMA	360.939	343.676	327.482	291.168	189.304	129.741	92.763	59.766	32.705	20.176
500	19.2241	CUSUM	486.626	461.144	437.247	383.728	235.376	151.703	102.263	61.059	30.789	18.538
	0.2725181	EWMA	486.394	460.637	436.678	383.657	240.346	160.496	112.564	71.008	37.950	23.040

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method (less than 1 second versus approximately 33 minutes, respectively). This means that the analytical IE for determining ARL_1 is a good alternative for detecting changes in the process mean on a CUSUM control chart.

A SARFIMAX process involving real data was run on both CUSUM and EWMA control charts. The performances of the control charts were compared in terms of the ARL for detecting small to moderate shifts in the process mean; the ARL constructed using the analytical IE was used on the CUSUM control chart whereas the one using the numerical IE method was used on the EWMA control chart [28]. For the performance comparison, boundary values $b = 17.37735, 19.2241$ for the CUSUM control chart and $b = 0.2507646, 0.2725181$ for the EWMA control chart were used with prespecified $ARL_0 = 370$ or $ARL_0 = 500$, and smoothing parameter λ for the EWMA control chart was determined as 0.1. The results of the comparison are summarized in Table 9. For $ARL_0 = 370$, the EWMA control chart provided a smaller ARL_1 than the CUSUM control chart for shift size $0 < \delta \leq 0.3$ whereas the CUSUM control chart provided a smaller ARL_1 than the EWMA control chart for shift size $0.5 < \delta \leq 2.0$. However, for $ARL_0 = 500$, the CUSUM control chart performed better than the EWMA control chart for shift sizes $0.3 < \delta \leq 2.0$. This means that the performance of the CUSUM control chart was more powerful than that of the EWMA chart when detecting moderate shifts in the process mean for $ARL_0 = 370$ and 500. Therefore, the CUSUM control chart was more efficacious than the EWMA control chart under these conditions, which is in accordance with the study results.

In Fig 3, the center line, the lower and upper control limits of the CUSUM control chart were $CL = \beta_0$, $LCL = 0$, $UCL = 17.37735$ and 19.2241 for $ARL_0 = 370$ and 500, respectively. We can see that the signal was first given at the 12th and 32nd-47th time points for $ARL_0 = 370$ and 500, respectively (the observations in red plotted above the upper control limit). These results confirm the sensitivity of the proposed analytical IE for the ARL on a CUSUM control chart providing 176 and 156 out-of-control signals for $ARL_0 = 370$ and 500, respectively. This situation directly affects the stock price of PPT. Hence, tracking changes in exchange rates can be key both in terms of economy and finance. In an investor's view, the control chart above is the point in making profitable decisions for investors.

Conclusions

Performance assessment of control chart can be measured by using the ARL determined via analytical and numerical IE methods used to analyze and approximate the ARL computation, respectively. We applied both methods to determine the ARL for monitoring changes in the mean of a long-memory SARFIMAX process with underlying exponential white noise on a CUSUM control chart. Computation of the ARL by using the numerical IE method was used to check the accuracy of the analytical IE method by using the percentage accuracy benchmark and comparing their computation times. The number of division points (m) for the numerical

IE method was large enough so that its ARL converged to that determined using the explicit formulas. The results show that the analytical IE was as accurate as the numerical IE method but required less computational time because its computational scheme was easier and could be completed in one stage.

The CUSUM control chart is suitable for detecting small and moderate parameter shifts for autocorrelated processes with bounded support [29, 30]. However, finding optimal parameters of the CUSUM control chart in these situations can be somewhat difficult. Therefore, obtaining the optimal values for the parameters of a CUSUM control chart should be studied further. The scope of the study could be extended in terms of the ARL and the methods used to compute it and could be applied to other types of processes including application to real data with other distributions of white noise such as gamma and Weibull from the exponential family.

Supporting information

S1 File.

(DOCX)

S1 Table.

(DOCX)

Author Contributions

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