

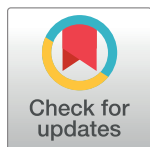
RESEARCH ARTICLE

Statistical analysis on finger replacement schemes for RAKE receivers in the soft handover region with multiple BSs over i.n.d. fading channels

Sung Sik Nam¹, Young-Chai Ko¹, Mohamed-Slim Alouini², Seyeong Choi^{3*}

1 School of Electrical Engineering, Korea University, Seoul, Korea, **2** Department of Electrical Engineering, KAUST, Thuwal, Makkah Province, Saudi Arabia, **3** Department of Information and Communication Engineering, Wonkwang University, Jeonbuk, Korea

* sychoi@wku.ac.kr



Abstract

A new finger replacement technique which is applicable for RAKE receivers in the soft handover region has been proposed and studied under the ideal assumption that the fading is both independent and identically distributed from path to path. To supplement our previous work, we present a general comprehensive framework for the performance assessment of the proposed finger replacement schemes operating over independent but non-identically distributed (i.n.d.) faded paths. To accomplish this object, we derive new closed-form expressions for the target key statistics which are composed of i.n.d. exponential random variables. With these new expressions, the performance analysis of various wireless communication systems over more practical channel environments can be possible.

OPEN ACCESS

Citation: Nam SS, Ko Y-C, Alouini M-S, Choi S (2017) Statistical analysis on finger replacement schemes for RAKE receivers in the soft handover region with multiple BSs over i.n.d. fading channels. PLoS ONE 12(6): e0179126. <https://doi.org/10.1371/journal.pone.0179126>

Editor: Yilun Shang, Tongji University, CHINA

Received: July 14, 2016

Accepted: May 24, 2017

Published: June 12, 2017

Copyright: © 2017 Nam et al. This is an open access article distributed under the terms of the [Creative Commons Attribution License](https://creativecommons.org/licenses/by/4.0/), which permits unrestricted use, distribution, and reproduction in any medium, provided the original author and source are credited.

Data Availability Statement: All relevant data are within the paper.

Funding: This paper was supported by Wonkwang University in 2017. The funders had no role in study design, data collection and analysis, decision to publish, or preparation of the manuscript.

Competing interests: The authors have declared that no competing interests exist.

Introduction

Multipath fading is an unavoidable physical phenomenon that affects considerably the performance of wideband wireless communication systems. While usually viewed as a deteriorating factor, multipath fading can also be exploited to improve the performance by using RAKE-type receivers. RAKE receiver is designed to optimally detect a signal transmitted over a dispersive multi-path channel. It is an extension of the concept of the matched filter. In the RAKE receiver, one RAKE finger is assigned to each multi-path, thus maximizing the amount of received signal energy. Each of these different paths are combined to form a composite signal that is expected to have substantially better characteristics for the purpose of demodulation than just a single path. However, in the soft handover (SHO) region, due to the limited number of fingers in the mobile unit, we are faced with a problem of how to judiciously select a subset of paths for RAKE reception to achieve the required performance.

Finger replacement techniques for RAKE reception in the SHO region have been proposed and analyzed over independent and identically distributed (i.i.d.) fading environments with two base stations (BSs) in [1] which was extended to the case of multiple BSs in [2]. The

proposed schemes in [2] are basically based on the block comparison among groups of resolvable paths from different BSs and lead to the reduction of complexity while offering commensurate performance in comparison with previously proposed schemes in [3, 4]. However, in practice, the i.i.d. fading scenario on the diversity paths is not always realistic due to, for example, the different adjacent multipath routes with the same path loss and the resulting unbalance among paths. This non-identical scenario may induce non-negligible performance degradation compared with the results for i.i.d. fading scenario and it may eventually lead to affect in determining system parameters (e.g., the number of combined paths). Although this non-identical consideration is important from a practical standpoint, [2] was able to investigate the effect of the non-uniform power delay profile of the finger replacement schemes only with computer simulations due to the high complexity of the analysis. The major difficulties lie in deriving the target statistics with non-identical parameters. Although the applied conditional probability density function (PDF) based approach used in [5] for deriving the required key statistics is quite impressive and contributive, the analytical framework in [5] was limited to i.i.d. Rayleigh fading assumptions.

With this observation in mind, in this work we mathematically attack these main difficulties. More specifically, we address the key mathematical formalisms which are the statistics of partial sums and the two-dimensional joint statistics of partial sums of the i.n.d. ordered random variables (RVs). Then, the mathematical analysis framework given in [2] is slightly modified suitable for these newly derived joint statistical results. Note that our derived results are much simpler than the original results with the multiple-fold integral forms based on the conventional moment generating function (MGF) based approaches. Note also that it is configured to be directly applicable to other applications and other fading scenarios.

It is very noticeable that the finger replacement schemes in [1, 2] can also apply to the “trendy” applications such as millimeter-wave (mmWave) communication systems [6–8] in which an antenna diversity scheme operates like a finger replacement scheme. In mmWave systems, with an increase of the number of Rake fingers, a significant improvement is expected because the channel impulse response is completely decayed in a very short time period compared with the typical RAKE receiver based systems (i.e., carrier frequencies below 10 GHz). Therefore, a larger number of fingers are required while there exist the limited number of fingers in the mobile unit. This can point to the very clear conclusion that it is more necessary to apply the low complexity and low power consumption finger management schemes with a minimal amount of additional network resources for RAKE reception in the SHO region with multiple base stations to achieve the required performance. In addition, for mmWave communication systems where antenna arrays are employed, the antenna elements may receive line-of-sight signals with different Rayleigh fading factor because these signals may take completely different (independent) propagation paths before arriving at the receiver [9]. Therefore, the non-identical condition assumed in this work is worth considering in terms of contribution.

Note that it is very important to accurately characterize the performance of wireless communication systems over more practical channel models as for examples in [10–12]. From this perspective, we believe that even though the overall analytical framework is based on the previous result in [2], the new results contained in this paper are quite suitable, general and contributive for the practical channel models in that the approach used in [2] even based on the i.i.d. fading scenario requires some case-specific manipulations.

The rest of this paper is organized as follows. In the ‘System Models and Performance Measures’ section, we present the system models as well as the mode of operation of the finger replacement scheme under consideration and provide the results of a general comprehensive framework for the outage performance based on the statistical results over i.n.d. fading

channels. We then provide in the ‘Key Statistics’ section, some closed-form expressions of the required key statistics. Final section provides some simulation results and concluding remarks.

System models and performance measures

Among the path scanning schemes proposed in [2], we consider the full scanning method. With this method, if the combined output signal-to-noise ratio (SNR) of current assigned fingers is greater than a certain target SNR, a one-way SHO is used and no finger replacement is needed. Otherwise, the receiver attempts a two-way SHO by starting to scan additional paths from the serving BS as well as all the target BSs.

We assume that L BSs are active, and there are a total of $N_{(L)}$ resolvable paths where $N_{(L)} =$

$\sum_{n=1}^L N_n$ and N_n is the number of resolvable paths from the n -th BS. In the SHO region as shown in Fig 1, only N_c out of $N_{(n)}$ ($1 \leq n \leq L$) paths are used for RAKE reception. Without loss of generality, let N_1 be the number of resolvable paths from the serving BS and N_2, N_3, \dots, N_L be those from the target BSs. In the SHO region, the receiver is assumed at first to rely only on N_1 resolvable paths and, as such, starts with N_c/N_1 -generalized selection combining (GSC) [13] which combines the strongest N_c resolvable paths among the N_1 available ones. These schemes are based on the comparison of blocks consisting of $N_s (< N_c < N_n)$ paths from each BS.

Let $u_{i,n}$ ($i = 1, 2, \dots, N_n$) be the i -th order statistics out of N_n SNRs of paths from the n -th BS by arranging N_n non-negative i.n.d. RVs, $\{\gamma_{j,n}\}_{j=1}^{N_n}$, where $\gamma_{j,n}$ is the SNR of the j -th path from the n -th BS, in decreasing order of magnitude such that $u_{1,n} \geq u_{2,n} \geq \dots \geq u_{N_n,n}$. If we let Y be the sum of the $N_c - N_s$ strongest paths from the serving BS and W_n be the sum of the N_s smallest paths from the serving BS for $n = 1$ and be the sum of the N_s strongest paths from the target BS for $n = 2, \dots, L$ as

$$Y = \sum_{i=1}^{N_c - N_s} u_{i,1} \quad (1)$$

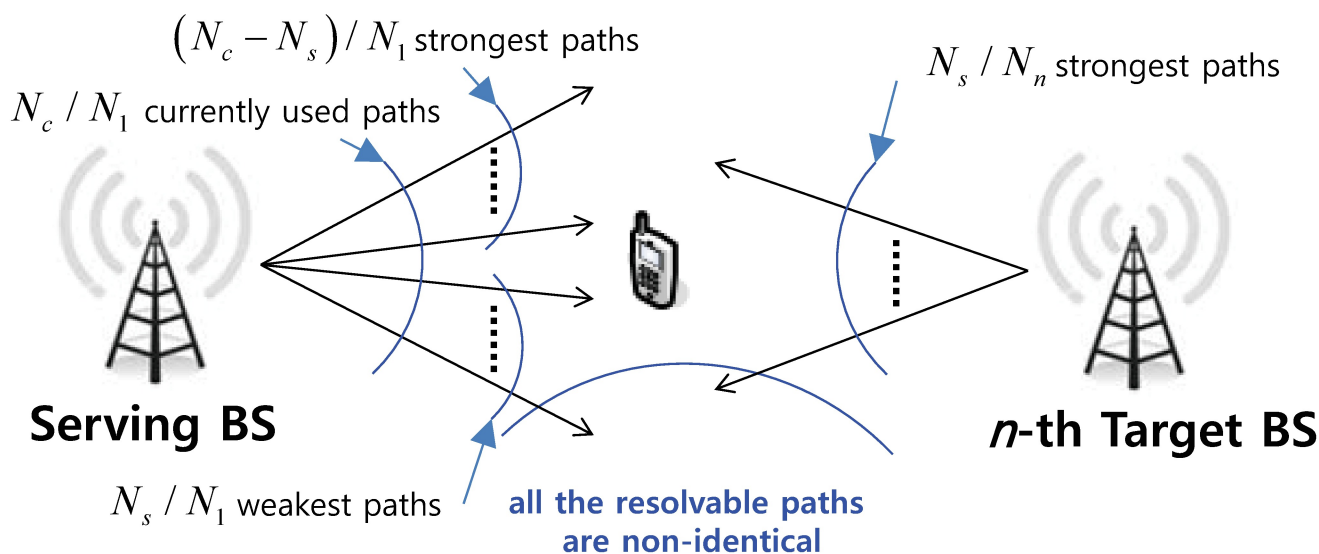


Fig 1. Finger replacement schemes for RAKE receivers in the soft handover region.

<https://doi.org/10.1371/journal.pone.0179126.g001>

and

$$W_n = \begin{cases} \sum_{i=N_c-N_s+1}^{N_c} u_{i,n}, & n = 1; \\ \sum_{i=1}^{N_s} u_{i,n}, & n = 2, \dots, L, \end{cases} \quad (2)$$

then, the received output SNR after GSC is given by $Y + W_1$. At the beginning of every time slot, the receiver compares the GSC output SNR, $Y + W_1$, with a certain target SNR. If $Y + W_1$ is greater than or equal to the target SNR, a one-way SHO is used and no finger replacement is needed. On the other hand, whenever $Y + W_1$ falls below the target SNR, the receiver attempts a two-way SHO by starting to scan additional paths from the target BSs.

To study the performance of the finger replacement scheme for i.n.d. fading assumptions, we look into the outage performance. Based on the mode of operation in [2], an overall outage probability is declared when the final combined SNR, γ_F , falls below a predetermined threshold, x , as

$$F_{\gamma_F}(x) = \Pr[\gamma_F < x] \quad (3)$$

where

$$\gamma_F = \begin{cases} Y + W_1, & Y + W_1 \geq \gamma_T; \\ Y + \max\{W_1, W_2, \dots, W_L\}, & Y + W_1 < \gamma_T. \end{cases} \quad (4)$$

Considering two cases that i) the final combined SNR is greater than or equal to the target SNR, γ_T , (i.e., $x \geq \gamma_T$) and ii) the final combined SNR falls below the target SNR, (i.e., $0 < x < \gamma_T$), separately, we can rewrite Eq (3) as

$$F_{\gamma_F}(x) = \begin{cases} \Pr[Y + \max\{W_1, W_2, \dots, W_L\} < x], & 0 < x < \gamma_T; \\ \Pr[\gamma_T \leq Y + W_1 < x] \\ + \Pr[Y + W_1 < \gamma_T, \gamma_T \leq Y + \max\{W_1, W_2, \dots, W_L\} < x], & x \geq \gamma_T. \end{cases} \quad (5)$$

Note that the major difficulty in the analysis is to derive the required key statistics of ordered RVs. In [1] and [2], with an i.i.d. assumption, the required statistics were obtained by applying the conditional PDF based approach proposed in [5]. Since our concerns are more practical channel models in which the average SNRs of each path (or branch) are different, unlike the i.i.d. case we need to consider realistic frequency selective channels which have a non-uniform delay profile (i.e., exponentially decaying power delay profile(PDP)). However, the proposed method in [5] can not be directly adopted in the case of i.n.d. fading environments here.

Recently, the unified framework to determine the joint statistics of partial sums of ordered i.i.d. RVs has been introduced in [14] which shows that the required key statistics of any partial sums of ordered RVs can be systematically obtained in terms of the MGF and the PDF. In [15, 16], some extended works from the mathematical approach proposed in [14] to i.n.d. fading channels can be found. In this paper, with the help of [14–16], the required key statistics to investigate the outage probability in Eq (5) over i.n.d. fading channels will be obtained.

Note that based on the mode of operation, Y and W_1 are correlated while W_n (for $n = 2, \dots, L$) is independent of Y . Hence, three probabilities in Eq (5) can be expressed as

$$\begin{aligned} & \Pr[Y + \max\{W_1, W_2, \dots, W_L\} < x] \\ &= \int_0^x \int_0^{x-y} \dots \int_0^{x-y} f_{Y, W_1, W_2, \dots, W_L}(y, w_1, w_2, \dots, w_L) dw_L \dots dw_2 dw_1 dy \\ &= \int_0^x \int_0^{x-y} \dots \int_0^{x-y} f_{Y, W_1}(y, w_1) f_{W_2}(w_2) \dots f_{W_L}(w_L) dw_L \dots dw_2 dw_1 dy \\ &= \int_0^x \int_0^{x-y} f_{Y, W_1}(y, w_1) \int_0^{x-y} f_{W_2}(w_2) dw_2 \dots \int_0^{x-y} f_{W_L}(w_L) dw_L \dots dw_1 dy \\ &= \int_0^x \int_0^{x-y} f_{Y, W_1}(y, w_1) \prod_{n=2}^L F_{W_n}(x-y) dw_1 dy, \end{aligned} \quad (6)$$

$$\Pr[\gamma_T \leq Y + W_1 < x] = F_{Y+W_1}(x) - F_{Y+W_1}(\gamma_T), \quad (7)$$

and

$$\begin{aligned} & \Pr[Y + W_1 < \gamma_T, \gamma_T \leq Y + \max\{W_1, W_2, \dots, W_L\} < x] \\ &= \int_0^{\gamma_T} \int_0^{\gamma_T-y} f_{Y, W_1}(y, w_1) \int_0^{x-y} f_{W_2}(w_2) dw_2 \dots \int_0^{x-y} f_{W_L}(w_L) dw_L \dots dw_1 dy \\ &= \int_0^{\gamma_T} \int_0^{\gamma_T-y} f_{Y, W_1}(y, w_1) \prod_{n=2}^L F_{W_n}(x-y) dw_1 dy. \end{aligned} \quad (8)$$

Similar to the identical case in [2], it is also very important to study the complexity of finger replacement schemes over i.n.d. case by accurately quantifying the performance measures such as the average number of path estimations, the average number of SNR comparisons, and the SHO overhead, which are required during the SHO process of these schemes over i.n.d. case. Note that with these performance measures, a comprehensive investigation of the trade-off between complexity and performance over i.n.d. fading channels can be feasible. These important design parameters can be evaluated by directly applying the defined formulas presented in [2] with the required key statistics for i.n.d. ordered RVs which will be derived in this work. Hence, based on the mathematical approach proposed in [14–16], we here focus on the derivation of the following key statistics such as the cumulative distribution function (CDF) [17] of the N_c/N_1 -GSC output SNR, $F_{Y+W_1}(\cdot)$, the 2-dimensional joint PDF of two adjacent partial sums, Y and W_1 , of order statistics, $f_{Y+W_1}(\cdot, \cdot)$, and the CDF of the sum of the N_s strongest paths from each target BS, $F_{W_n}(\cdot)$ ($2 \leq n \leq L$).

Key statistics

In this section, we introduce the key statistics which are essential to solve Eqs (6), (7), and (8). More specifically, in these three cases, only the best N_c or N_s among N_n ($N_s \leq N_c \leq N_n$) ordered RVs are involved in the partial sums. Thus, based on the unified frame work in [14] and the extended work for i.n.d. case in [15, 16], each key statistics for three cases can be derived by applying the special step approach based on the substituted groups instead of original groups for each cases (i.e., starting from 2-dimensional joint statistics, 4-dimensional joint statistics, and 2-dimensional joint statistics, respectively) as follows;

1. $F_{Y+W_1}(x)$: In this case, Y and W_1 have the following relationship

$$\overbrace{u_{1,1}, \dots, u_{N_c-N_s-1,1}, u_{N_c-N_s,1}}^Y, \overbrace{u_{N_c-N_s+1,1}, \dots, u_{N_c-1,1}, u_{N_c,1}}^{W_1}, u_{N_c+1,1}, \dots, u_{N,1}. \quad (9)$$

Here, by adopting the unified-frame work in [14], the target CDF of interest, $F_{Y+W_1}(x)$, can be obtained from the transformed higher dimensional joint PDFs where Z_1 and Z_2 have the following relationships

$$\underbrace{u_{1,1}, \dots, u_{N_c-N_s-1,1}, u_{N_c-N_s,1}, u_{N_c-N_s+1,1}, \dots, u_{N_c-1,1}}_{Z_1}, \underbrace{u_{N_c,1}}_{Z_2}, u_{N_c+1,1}, \dots, u_{N,1}. \quad (10)$$

Therefore, if we let $Z' = Y + W_1$ where $Y + W_1 = \sum_{i=1}^{N_c} u_{i,1}$ for convenience, then $Z_2 < \frac{Z'}{N_c}$ and

as a result, we can derive the target CDF of Z' with the 2-dimensional joint PDF of $Z_1 =$

$$\begin{aligned} \sum_{i=1}^{N_c-1} u_{i,1} \text{ and } Z_2 = u_{N_c,1} \text{ as} \\ F_{Y+W_1}(x) &= \int_0^x f_{Z'}(z) dz \\ &= \int_0^x \int_0^{\frac{z}{N_c}} f_{Z_1, Z_2}(z - z_2, z_2) dz_2 dz. \end{aligned} \quad (11)$$

2. $f_{Y+W_1}(x, y)$: In this case, we can derive the target 2-dimensional PDF of $Y = \sum_{i=1}^{N_c-N_s} u_{i,1}$ and

$$W_1 = \sum_{i=N_c-N_s+1}^{N_c} u_{i,1} \text{ by transferring the 4-dimensional joint PDF of } Z_1 = \sum_{i=1}^{N_c-N_s-1} u_{i,1},$$

$$Z_2 = u_{N_c-N_s,1}, Z_3 = \sum_{i=N_c-N_s+1}^{N_c-1} u_{i,1}, \text{ and } Z_4 = u_{N_c,1} \text{ where } Z_1, Z_2, Z_3, \text{ and } Z_4 \text{ have the follow-}$$

ing relationships

$$\underbrace{u_{1,1}, \dots, u_{N_c-N_s-1,1}}_{Z_1}, \underbrace{u_{N_c-N_s,1}}_{Z_2}, \underbrace{u_{N_c-N_s+1,1}, \dots, u_{N_c-1,1}}_{Z_3}, \underbrace{u_{N_c,1}}_{Z_4}, u_{N_c+1,1}, \dots, u_{N,1}. \quad (12)$$

From Eq (12), the following valid conditions can be directly obtained

$$Z_1 \geq (N_c - N_s - 1)Z_2 \Rightarrow Z_1 + Z_2 \geq (N_c - N_s)Z_2, \quad (13)$$

$$Z_3 \geq (N_s - 1)Z_4 \Rightarrow Z_3 + Z_4 \geq N_s \cdot Z_4, \quad (14)$$

and

$$N_s \cdot Z_2 \geq Z_3 + Z_4. \quad (15)$$

Therefore, with the 4-dimensional joint PDF of Z_1, Z_2, Z_3 , and Z_4 , letting $X = Z_1 + Z_2$ and $Y = Z_3 + Z_4$, we can obtain the target 2-dimensional joint PDF of Y and W_1 with the help of

a function of a marginal PDF by integrating over z_2 and z_4 yielding

$$f_{Y,W_1}(x, y) = \int_0^{\frac{y}{N_s}} \int_{\frac{y}{N_s}}^{\frac{x}{N_s - N_s}} f_{Z_1, Z_2, Z_3, Z_4}(x - z_2, z_2, y - z_4, z_4) dz_2 dz_4. \quad (16)$$

3. $F_{W_n}(x)$ ($2 \leq n \leq L$): Similar to case 1), the target one-dimensional CDF of $W_n = \sum_{i=1}^{N_s} u_{i,n}$

with the 2-dimensional joint PDF of $Z'_1 = \sum_{i=1}^{N_s-1} u_{i,n}$ and $Z'_2 = u_{N_s,n}$ can be derived with the help of a function of a marginal PDF as

$$F_{W_n}(x) = \int_0^x \int_0^{\frac{x}{N_s}} f_{Z'_1, Z'_2}(z - z'_2, z'_2) dz'_2 dz. \quad (17)$$

Note that the above novel generic results in Eqs (11), (16) and (17) are quite general and can be applied for any RVs. In here, we assume the i.n.d. RVs with a common exponential PDF and CDF given by

$$p_{i,n}(x) = \frac{1}{\bar{\gamma}_{i,n}} \exp\left(-\frac{x}{\bar{\gamma}_{i,n}}\right), \quad x \geq 0 \quad (18)$$

and

$$P_{i,n}(x) = 1 - \exp\left(-\frac{x}{\bar{\gamma}_{i,n}}\right), \quad x \geq 0 \quad (19)$$

respectively, where $\bar{\gamma}_{i,n}$ is the average of the i -th RV at n -th BS. In what follows, for convenience in a mathematical representation, we define the set of $(N_n - m - 1)$ -tuples, $\tau_n(i, m, N_n)$, as $\tau_n(i, m, N_n) = \{i_{m,n}, i_{m+1,n}, \dots, i_{N_n,n}\}$. We also assume, without loss of generality, that $N_n = N$, $u_{i,n} = u_i$, and $\bar{\gamma}_{i,n} = \bar{\gamma}_i$ for all $n = 1, 2, \dots, L$. For the analytical tractability, we start from $F_{Y,W_1}(x, y)$.

Joint PDF of two adjacent partial sums Y and W_1 over i.n.d. Rayleigh fading, $f_{Y,W_1}(x, y)$

As shown in Eq (16), the target 2-dimensional joint PDF of interest, $f_{Y,W_1}(x, y)$, can be obtained from the transformed higher dimensional joint PDF. Based on the result in [16, Eq (54)], to evaluate the 2-fold integrations in Eq (16), the multiple product expression,

$\prod_{j=N_s+1}^N \left(1 - \exp\left(-\frac{z_4}{\bar{\gamma}_{ij}}\right)\right)$, needs to be converted to the summation expression with the help of

the property of exponential multiplication. Note that the product of two exponential numbers of the same base can be simply represented as the sum of the exponents with the same base.

Let $N_s + 1 = n_1$, $N = n_2$, and $\exp\left(-\frac{z_4}{\bar{\gamma}_{ij}}\right) = \exp(-a_{ij})$ for convenience, then

$$\prod_{j=N_s+1}^N \left(1 - \exp\left(-\frac{z_4}{\bar{\gamma}_{ij}}\right)\right) = \prod_{j=n_1}^{n_2} (1 - \exp(-a_{ij})). \quad (20)$$

Starting from the case of $n_1 = 1$ and $n_2 = 2$, we extend this result to the general case for arbitrary n_1 and n_2 . At first, let $n_1 = 1$ and $n_2 = 2$, then we can write Eq (20) as the following

summation expression

$$\begin{aligned} \prod_{j=1}^2 (1 - \exp(-a_{i_j})) &= (1 - \exp(-a_{i_1}))(1 - \exp(-a_{i_2})) \\ &= 1 - \exp(-a_{i_1}) - \exp(-a_{i_2}) + \exp(-a_{i_1} - a_{i_2}). \end{aligned} \quad (21)$$

Similarly, for $n_1 = 1$ and $n_2 = 3$, we get

$$\begin{aligned} \prod_{j=1}^3 (1 - \exp(-a_{i_j})) &= (1 - \exp(-a_{i_1}))(1 - \exp(-a_{i_2}))(1 - \exp(-a_{i_3})) \\ &= 1 - \exp(-a_{i_1}) - \exp(-a_{i_2}) - \exp(-a_{i_3}) + \exp(-a_{i_1} - a_{i_2}) + \exp(-a_{i_1} - a_{i_3}) \\ &\quad + \exp(-a_{i_2} - a_{i_3}) - \exp(-a_{i_1} - a_{i_2} - a_{i_3}). \end{aligned} \quad (22)$$

After simplification with a few manipulations, we can re-write the multiple product expressions in Eqs (21) and (22) as the following simplified summation expressions, respectively,

$$\begin{aligned} \prod_{j=1}^2 (1 - \exp(-a_{i_j})) \\ = 1 + (-1) \sum_{j'_1=1}^2 \exp(-a_{i_{j'_1}}) + (-1)^2 \sum_{j'_1=1}^1 \sum_{j'_2=j'_1+1}^2 \exp(-a_{i_{j'_1}} - a_{i_{j'_2}}), \end{aligned} \quad (23)$$

and

$$\begin{aligned} \prod_{j=1}^3 (1 - \exp(-a_{i_j})) \\ = 1 + (-1) \sum_{j'_1=1}^3 \exp(-a_{i_{j'_1}}) + (-1)^2 \sum_{j'_1=1}^2 \sum_{j'_2=j'_1+1}^3 \exp(-a_{i_{j'_1}} - a_{i_{j'_2}}) \\ + (-1)^3 \sum_{j'_1=1}^1 \sum_{j'_2=j'_1+1}^2 \sum_{j'_3=j'_2+1}^3 \exp(-a_{i_{j'_1}} - a_{i_{j'_2}} - a_{i_{j'_3}}). \end{aligned} \quad (24)$$

Finally, after simplifying and generalizing the above equations, we can obtain the generalized expression of Eq (20) as

$$\begin{aligned} \prod_{j=N_s+1}^N \left(1 - \exp\left(-\frac{z_4}{\gamma_{i_j}}\right) \right) \\ = 1 + \sum_{g=1}^{N-N_s} (-1)^g \sum_{j'_1=j'_0+N_s+1}^{N-g+1} \cdots \sum_{j'_g=j'_{g-1}+1}^N \exp\left(-\sum_{m=1}^g \frac{z_4}{\gamma_{i_{j'_m}}}\right). \end{aligned} \quad (25)$$

Note that with the help of Eq (25), the 4-dimensional joint PDF given in [16, Eq (45)] can be expressed in the summation form which makes Eq (16) more tractable. Hence, inserting the rewritten expression of [16, Eq (45)] into Eq (16), after some manipulations, we can rewrite

Eq (16) as

$$\begin{aligned}
 f_{Y,W_1}(x,y) = & \sum_{\substack{1,2,\dots,N_1 \\ i_{N_c}, \dots, i_{N_1} \\ i_{N_c} \neq \dots \neq i_{N_1}}} \frac{1}{\bar{\gamma}_{i_{N_c}}} \sum_{\substack{N_1 \\ i_{N_c-N_s} = 1 \\ i_{N_c-N_s} \neq i_{N_c}, \dots, i_{N_1}}} \frac{1}{\bar{\gamma}_{i_{N_c-N_s}}} \sum_{\substack{\{i_{N_c-N_s+1}, \dots, i_{N_c-1}\} \in \\ \mathcal{P}_{N_s-1}(I_{N_1} - \{i_{N_c-N_s}\} - \{i_{N_c}, \dots, i_{N_1}\})}} \sum_{k=N_c-N_s+1}^{N_c-1} C_{k,N_c-N_s+1,N_c-1} \\
 & \left[\sum_{\substack{\{i_1, \dots, i_{N_c-N_s-1}\} \in \\ \mathcal{P}_{N_c-N_s-1}(I_{N_1} - \{i_{N_c-N_s}\} - \{i_{N_c}, \dots, i_{N_1}\} - \{i_{N_c-N_s+1}, \dots, i_{N_c-1}\})}} \sum_{h=1}^{N_c-N_s-1} C_{h,1,N_c-N_s-1} \exp\left(-\frac{x}{\bar{\gamma}_{i_h}}\right) \exp\left(-\frac{y}{\bar{\gamma}_{i_k}}\right) \right. \\
 & \times \int_0^{\frac{x}{N_s}} \int_{\frac{y}{N_s}}^{\frac{x-N_s}{N_s}} \exp\left(-\left(\sum_{l=1}^{N_c-N_s} \left(\frac{1}{\bar{\gamma}_{i_l}}\right) - \frac{N_c-N_s}{\bar{\gamma}_{i_h}}\right) z_2\right) \exp\left(-\left(\sum_{l=N_c-N_s+1}^{N_c} \left(\frac{1}{\bar{\gamma}_{i_l}}\right) - \frac{(N_s)}{\bar{\gamma}_{i_k}}\right) z_4\right) U(x - (N_c - N_s) \cdot z_2) U(y - (N_s) \cdot z_4) dz_2 dz_4 \\
 & + \sum_{l=1}^{N_c-1} (-1)^l \sum_{j_1=j_0+N_c-N_s+1}^{N_c-l} \dots \sum_{j_l=j_{l-1}+1}^{N_c-1} \sum_{\substack{\{i_1, \dots, i_{N_c-N_s-1}\} \in \\ \mathcal{P}_{N_c-N_s-1}(I_{N_1} - \{i_{N_c-N_s}\} - \{i_{N_c}, \dots, i_{N_1}\} - \{i_{N_c-N_s+1}, \dots, i_{N_c-1}\})}} \sum_{h=1}^{N_c-N_s-1} C_{h,1,N_c-N_s-1} \exp\left(-\frac{x}{\bar{\gamma}_{i_h}}\right) \exp\left(-\frac{y}{\bar{\gamma}_{i_k}}\right) \\
 & \times \int_0^{\frac{x}{N_s}} \int_{\frac{y}{N_s}}^{\frac{x-N_s}{N_s}} \exp\left(-\left(\sum_{l=1}^{N_c-N_s} \left(\frac{1}{\bar{\gamma}_{i_l}}\right) + \sum_{q=1}^l \frac{1}{\bar{\gamma}_{i_{j_q}}} - \frac{N_c-N_s}{\bar{\gamma}_{i_h}} - \frac{l}{\bar{\gamma}_{i_k}}\right) z_2\right) \exp\left(-\left(\sum_{l=N_c-N_s+1}^{N_c} \left(\frac{1}{\bar{\gamma}_{i_l}}\right) - \frac{(N_s-l)}{\bar{\gamma}_{i_k}} - \sum_{q=1}^l \frac{1}{\bar{\gamma}_{i_{j_q}}}\right) z_4\right) \\
 & U(x - (N_c - N_s) \cdot z_2) U(y - (l \cdot z_2 + (N_s - l) \cdot z_4)) dz_2 dz_4 \Big] \\
 & + \sum_{\substack{1,2,\dots,N_1 \\ i_{N_c}, \dots, i_{N_1} \\ i_{N_c} \neq \dots \neq i_{N_1}}} \frac{1}{\bar{\gamma}_{i_{N_c}}} \sum_{g=1}^{N_1-N_c} (-1)^g \sum_{j_1=j_0+N_c+1}^{N_1-g+1} \dots \sum_{j_g=j_{g-1}+1}^{N_1} \sum_{\substack{N_1 \\ i_{N_c-N_s} = 1 \\ i_{N_c-N_s} \neq i_{N_c}, \dots, i_{N_1}}} \frac{1}{\bar{\gamma}_{i_{N_c-N_s}}} \\
 & \sum_{\substack{\{i_{N_c-N_s+1}, \dots, i_{N_c-1}\} \in \\ \mathcal{P}_{N_s-1}(I_{N_1} - \{i_{N_c-N_s}\} - \{i_{N_c}, \dots, i_{N_1}\})}} \sum_{k=N_c-N_s+1}^{N_c-1} C_{k,N_c-N_s+1,N_c-1} \\
 & \left[\sum_{\substack{\{i_1, \dots, i_{N_c-N_s-1}\} \in \\ \mathcal{P}_{N_c-N_s-1}(I_{N_1} - \{i_{N_c-N_s}\} - \{i_{N_c}, \dots, i_{N_1}\} - \{i_{N_c-N_s+1}, \dots, i_{N_c-1}\})}} \sum_{h=1}^{N_c-N_s-1} C_{h,1,N_c-N_s-1} \exp\left(-\frac{x}{\bar{\gamma}_{i_h}}\right) \exp\left(-\frac{y}{\bar{\gamma}_{i_k}}\right) \right. \\
 & \times \int_0^{\frac{x}{N_s}} \int_{\frac{y}{N_s}}^{\frac{x-N_s}{N_s}} \exp\left(-\left(\sum_{l=1}^{N_c-N_s} \left(\frac{1}{\bar{\gamma}_{i_l}}\right) - \frac{N_c-N_s}{\bar{\gamma}_{i_h}}\right) z_2\right) \exp\left(-\left(\sum_{l=N_c-N_s+1}^{N_c} \left(\frac{1}{\bar{\gamma}_{i_l}}\right) + \sum_{m=1}^g \frac{1}{\bar{\gamma}_{i_{j_m}}} - \frac{(N_s)}{\bar{\gamma}_{i_k}}\right) z_4\right) \\
 & U(x - (N_c - N_s) \cdot z_2) U(y - (N_s) \cdot z_4) dz_2 dz_4 \\
 & + \sum_{l=1}^{N_c-1} (-1)^l \sum_{j_1=j_0+N_c-N_s+1}^{N_c-l} \dots \sum_{j_l=j_{l-1}+1}^{N_c-1} \sum_{\substack{\{i_1, \dots, i_{N_c-N_s-1}\} \in \\ \mathcal{P}_{N_c-N_s-1}(I_{N_1} - \{i_{N_c-N_s}\} - \{i_{N_c}, \dots, i_{N_1}\} - \{i_{N_c-N_s+1}, \dots, i_{N_c-1}\})}} \sum_{h=1}^{N_c-N_s-1} C_{h,1,N_c-N_s-1} \exp\left(-\frac{x}{\bar{\gamma}_{i_h}}\right) \exp\left(-\frac{y}{\bar{\gamma}_{i_k}}\right) \\
 & \times \int_0^{\frac{x}{N_s}} \int_{\frac{y}{N_s}}^{\frac{x-N_s}{N_s}} \exp\left(-\left(\sum_{l=1}^{N_c-N_s} \left(\frac{1}{\bar{\gamma}_{i_l}}\right) + \sum_{q=1}^l \frac{1}{\bar{\gamma}_{i_{j_q}}} - \frac{N_c-N_s}{\bar{\gamma}_{i_h}} - \frac{l}{\bar{\gamma}_{i_k}}\right) z_2\right) \exp\left(-\left(\sum_{l=N_c-N_s+1}^{N_c} \left(\frac{1}{\bar{\gamma}_{i_l}}\right) + \sum_{m=1}^g \frac{1}{\bar{\gamma}_{i_{j_m}}} - \frac{(N_s-l)}{\bar{\gamma}_{i_k}} - \sum_{q=1}^l \frac{1}{\bar{\gamma}_{i_{j_q}}}\right) z_4\right) \\
 & U(x - (N_c - N_s) \cdot z_2) U(y - (l \cdot z_2 + (N_s - l) \cdot z_4)) dz_2 dz_4 \Big].
 \end{aligned} \tag{26}$$

In Eq (26), the closed-form expressions for the first and the third integral terms can be obtained by simply applying the basic exponential integration [18] and using the following useful common function

$$\begin{aligned} I(x, e, a, b; y, f, c, d) &= \int_c^d \int_a^b \exp(e \cdot x) \exp(f \cdot y) dx dy \\ &= \frac{1}{e \cdot f} \{ \exp(e \cdot b) - \exp(e \cdot a) \} \{ \exp(f \cdot d) - \exp(f \cdot c) \}. \end{aligned} \quad (27)$$

With Eq (27), letting $\alpha = -\left(\sum_{l=N_c-N_s+1}^{N_c} \left(\frac{1}{\tilde{\gamma}_{il}}\right) - \frac{(N_s)}{\tilde{\gamma}_{ik}}\right)$ and $\beta = -\left(\sum_{l=1}^{N_c-N_s} \left(\frac{1}{\tilde{\gamma}_{il}}\right) - \frac{N_c-N_s}{\tilde{\gamma}_{ih}}\right)$, we can express the first integral term in Eq (26) as

$$\begin{aligned} &\int_0^y \int_{\frac{y}{N_s}}^{\frac{x}{N_c-N_s}} \exp(\beta z_2) \exp(\alpha z_4) U(x - (N_c - N_s) \cdot z_2) U(y - (N_s) \cdot z_4) dz_2 dz_4 \\ &= \int_0^y \int_{\frac{y}{N_s}}^{\frac{x}{N_c-N_s}} \exp(\beta z_2) \exp(\alpha z_4) dz_2 dz_4 \\ &= I\left(z_2, \beta, \frac{y}{N_s}, \frac{x}{N_c - N_s}; z_4, \alpha, 0, \frac{y}{N_s}\right). \end{aligned} \quad (28)$$

Similarly, by replacing α with α' in Eq (28) where $\alpha' = -\left(\sum_{l=N_c-N_s+1}^{N_c} \left(\frac{1}{\tilde{\gamma}_{il}}\right) + \sum_{m=1}^g \frac{1}{\tilde{\gamma}_{lm}} - \frac{(N_s)}{\tilde{\gamma}_{ik}}\right)$, we can easily obtain the third integral term in Eq (26).

For the second and fourth integral terms, we need a special care on the valid integration region of z_2 and z_4 . More specifically, z_2 should satisfy two following conditions

$$z_2 \leq \frac{x}{N_c - N_s} \quad \text{and} \quad z_2 \leq \frac{y - (N_s - l)z_4}{l} \quad (29)$$

which directly lead

$$z_2 \leq \min\left[\frac{x}{N_c - N_s}, \frac{y - (N_s - l)z_4}{l}\right]. \quad (30)$$

Hence, if $\frac{x}{N_c - N_s} \leq \frac{y - (N_s - l)z_4}{l}$, we get

$$\frac{y}{N_s} < z_2 < \frac{x}{N_c - N_s} \quad \text{and} \quad 0 < z_4 < \min\left[\frac{y}{N_s}, \frac{y - \frac{l}{N_c - N_s}x}{(N_s - l)}\right]. \quad (31)$$

Otherwise, if $\frac{x}{N_c - N_s} > \frac{y - (N_s - l)z_4}{l}$, the valid integral regions for z_2 is

$$\frac{y}{N_s} < z_2 < \frac{y - (N_s - l)z_4}{l}, \quad (32)$$

and for z_4 , we need to consider both cases

$$0 < z_4 < \min\left[\frac{y}{N_s}, \frac{y - \frac{l}{N_c - N_s}x}{(N_s - l)}\right] \quad \text{and} \quad 0 < z_4 < \frac{y}{N_s} \quad (33)$$

by considering the unit step function, $\{1 - U\left(\frac{x}{N_c - N_s} - \frac{y - (N_s - l)z_4}{l}\right)\}$. As results, we can express

the second integral term in Eq (26) as three double-integral terms

$$\begin{aligned} & \int_0^{\min\left[\frac{y}{N_s}, \frac{y - \frac{l}{N_c - N_s} \cdot x}{(N_s - l)}\right]} \exp\left(-\left(\sum_{l=N_c - N_s + 1}^{N_c} \left(\frac{1}{\bar{\gamma}_{i_l}}\right) - \frac{(N_s - l)}{\bar{\gamma}_{i_k}} - \sum_{q=1}^l \frac{1}{\bar{\gamma}_{i_{lq}}}\right) z_4\right) \\ & \int_{\frac{y}{N_s}}^{\frac{x}{N_c - N_s}} \exp\left(-\left(\sum_{l=1}^{N_c - N_s} \left(\frac{1}{\bar{\gamma}_{i_l}}\right) + \sum_{q=1}^l \frac{1}{\bar{\gamma}_{i_{lq}}} - \frac{N_c - N_s}{\bar{\gamma}_{i_h}} - \frac{l}{\bar{\gamma}_{i_k}}\right) z_2\right) dz_2 dz_4 \\ & + \left\{ \int_0^{\frac{y}{N_s}} \exp\left(-\left(\sum_{l=N_c - N_s + 1}^{N_c} \left(\frac{1}{\bar{\gamma}_{i_l}}\right) - \frac{(N_s - l)}{\bar{\gamma}_{i_k}} - \sum_{q=1}^l \frac{1}{\bar{\gamma}_{i_{lq}}}\right) z_4\right) \right. \\ & \int_{\frac{y}{N_s}}^{\frac{y - (N_s - l) \cdot z_4}{l}} \exp\left(-\left(\sum_{l=1}^{N_c - N_s} \left(\frac{1}{\bar{\gamma}_{i_l}}\right) + \sum_{q=1}^l \frac{1}{\bar{\gamma}_{i_{lq}}} - \frac{N_c - N_s}{\bar{\gamma}_{i_h}} - \frac{l}{\bar{\gamma}_{i_k}}\right) z_2\right) dz_2 dz_4 \\ & - \int_0^{\min\left[\frac{y}{N_s}, \frac{y - \frac{l}{N_c - N_s} \cdot x}{(N_s - l)}\right]} \exp\left(-\left(\sum_{l=N_c - N_s + 1}^{N_c} \left(\frac{1}{\bar{\gamma}_{i_l}}\right) - \frac{(N_s - l)}{\bar{\gamma}_{i_k}} - \sum_{q=1}^l \frac{1}{\bar{\gamma}_{i_{lq}}}\right) z_4\right) \\ & \left. \int_{\frac{y}{N_s}}^{\frac{y - (N_s - l) \cdot z_4}{l}} \exp\left(-\left(\sum_{l=1}^{N_c - N_s} \left(\frac{1}{\bar{\gamma}_{i_l}}\right) + \sum_{q=1}^l \frac{1}{\bar{\gamma}_{i_{lq}}} - \frac{N_c - N_s}{\bar{\gamma}_{i_h}} - \frac{l}{\bar{\gamma}_{i_k}}\right) z_2\right) dz_2 dz_4 \right\}. \end{aligned} \quad (34)$$

The fourth integral term in Eq (26) can be also expressed in a quite similar way.

In Eq (34), the closed-form expression of the first double-integral term can be obtained with the help of Eq (27). However, for the second and the third double-integral terms in Eq (34), the inner integral limit depends on the outer variable. Therefore, for these cases, we can obtain the closed-form expression with the help of another useful common function as

$$\begin{aligned} & \Gamma'(x, e, a, b - b'y; y, f, c, d) \\ & = \int_c^d \exp(f \cdot y) \int_a^{b - b'y} \exp(e \cdot x) dx dy \\ & = \frac{1}{e} \left[\exp(e \cdot a) \frac{1}{(f - e \cdot b')} \{ \exp((f - e \cdot b') \cdot d) - \exp((f - e \cdot b') \cdot c) \} \right. \\ & \quad \left. - \exp(e \cdot a) \frac{1}{f} \{ \exp(f \cdot d) - \exp(f \cdot c) \} \right]. \end{aligned} \quad (35)$$

Letting $\alpha'' = -\left(\sum_{l=N_c - N_s + 1}^{N_c} \left(\frac{1}{\bar{\gamma}_{i_l}}\right) - \frac{(N_s - l)}{\bar{\gamma}_{i_k}} - \sum_{q=1}^l \frac{1}{\bar{\gamma}_{i_{lq}}}\right)$ and $\beta' = -\left(\sum_{l=1}^{N_c - N_s} \left(\frac{1}{\bar{\gamma}_{i_l}}\right) + \sum_{q=1}^l \frac{1}{\bar{\gamma}_{i_{lq}}} - \frac{N_c - N_s}{\bar{\gamma}_{i_h}} - \frac{l}{\bar{\gamma}_{i_k}}\right)$, we can finally obtain the closed-form expression of Eq (34) or equivalently the second integral term in Eq (26) as

$$\begin{aligned} & \Gamma\left(z_2, \beta', \frac{y}{N_s}, \frac{x}{N_c - N_s}; z_4, \alpha'', 0, \min\left[\frac{y}{N_s}, \frac{y - \frac{l}{N_c - N_s} \cdot x}{(N_s - l)}\right]\right) \\ & + \Gamma\left(z_2, \beta', \frac{y}{N_s}, \frac{y - (N_s - l) \cdot z_4}{l}; z_4, \alpha'', 0, \frac{y}{N_s}\right) \\ & - \Gamma\left(z_2, \beta', \frac{y}{N_s}, \frac{y - (N_s - l) \cdot z_4}{l}; z_4, \alpha'', 0, \min\left[\frac{y}{N_s}, \frac{y - \frac{l}{N_c - N_s} \cdot x}{(N_s - l)}\right]\right). \end{aligned} \quad (36)$$

For fourth integral term in Eq (26), by replacing α'' with α''' in Eq (36) where

$$\alpha''' = - \left(\sum_{l=N_c-N_s+1}^{N_c} \left(\frac{1}{\bar{\gamma}_{i_l}} \right) + \sum_{m=1}^g \frac{1}{\bar{\gamma}_{i'_m}} - \frac{(N_s-l)}{\bar{\gamma}_{i_k}} - \sum_{q=1}^l \frac{1}{\bar{\gamma}_{i_{jq}}} \right),$$

we can easily obtain the closed-form result.

Then, by substituting all these closed-form results and after some manipulations, we can obtain the final closed-form expression of Eq (16) as

$$\begin{aligned} f_{Y,W_1}(x,y) = & \sum_{\substack{i_{N_c,1}, \dots, i_{N_1,1} \\ i_{N_c,1} \neq \dots \neq i_{N_1,1}}}^{1,2,\dots,N_1} \frac{1}{\bar{\gamma}_{i_{N_c,1}}} \sum_{\substack{i_{N_c-N_s,1} = 1 \\ i_{N_c-N_s,1} \neq i_{N_c,1}, \dots, i_{N_1,1}}}^{N_1} \frac{1}{\bar{\gamma}_{i_{N_c-N_s,1}}} \sum_{\tau_{B,3} \in} \sum_{k=N_c-N_s+1}^{N_c-1} C_{k,N_c-N_s+1,N_c-1} \\ & \left[\sum_{\tau_{B,1} \in} \sum_{h=1}^{N_c-N_s-1} C_{h,1,N_c-N_s-1} \exp \left(- \left(\frac{x}{\bar{\gamma}_{i_{h,1}}} + \frac{y}{\bar{\gamma}_{i_{k,1}}} \right) \right) I \left(z_2, \beta, \frac{y}{N_s}, \frac{x}{N_c-N_s}; z_4, \alpha, 0, \frac{y}{N_s} \right) \right. \\ & + \sum_{l=1}^{N_s-1} (-1)^l \sum_{j_1=j_0+N_c-N_s+1}^{N_c-l} \dots \sum_{j_l=j_{l-1}+1}^{N_c-1} \sum_{\tau_{B,1} \in} \sum_{h=1}^{N_c-N_s-1} C_{h,1,N_c-N_s-1} \exp \left(- \left(\frac{x}{\bar{\gamma}_{i_{h,1}}} + \frac{y}{\bar{\gamma}_{i_{k,1}}} \right) \right) \\ & \times \left\{ I \left(z_2, \beta', \frac{y}{N_s}, \frac{x}{N_c-N_s}; z_4, \alpha'', 0, \min \left[\frac{y}{N_s}, \frac{y - \frac{l}{N_c-N_s} \cdot x}{(N_s-l)} \right] \right) + I' \left(z_2, \beta', \frac{y}{N_s}, \frac{y - (N_s-l) \cdot z_4}{l}; z_4, \alpha'', 0, \frac{y}{N_s} \right) \right. \\ & \left. \left. - I' \left(z_2, \beta', \frac{y}{N_s}, \frac{y - (N_s-l) \cdot z_4}{l}; z_4, \alpha'', 0, \min \left[\frac{y}{N_s}, \frac{y - \frac{l}{N_c-N_s} \cdot x}{(N_s-l)} \right] \right) \right\} \right] \\ & + \sum_{\substack{i_{N_c,1}, \dots, i_{N_1,1} \\ i_{N_c,1} \neq \dots \neq i_{N_1,1}}}^{1,2,\dots,N_1} \frac{1}{\bar{\gamma}_{i_{N_c,1}}} \sum_{g=1}^{N_1-N_c} (-1)^g \sum_{j'_1=j'_0+N_c+1}^{N_1-g+1} \dots \sum_{j'_g=j'_{g-1}+1}^{N_1} \sum_{\substack{i_{N_c-N_s,1} = 1 \\ i_{N_c-N_s,1} \neq i_{N_c,1}, \dots, i_{N_1,1}}}^{N_1} \frac{1}{\bar{\gamma}_{i_{N_c-N_s,1}}} \sum_{\tau_{B,3} \in} \\ & \left[\sum_{k=N_c-N_s+1}^{N_c-1} C_{k,N_c-N_s+1,N_c-1} \sum_{\tau_{B,1} \in} \sum_{h=1}^{N_c-N_s-1} C_{h,1,N_c-N_s-1} \exp \left(- \left(\frac{x}{\bar{\gamma}_{i_{h,1}}} + \frac{y}{\bar{\gamma}_{i_{k,1}}} \right) \right) \right. \\ & \times I \left(z_2, \beta, \frac{y}{N_s}, \frac{x}{N_c-N_s}; z_4, \alpha', 0, \frac{y}{N_s} \right) \\ & + \sum_{l=1}^{N_s-1} (-1)^l \sum_{j_1=j_0+N_c-N_s+1}^{N_c-l} \dots \sum_{j_l=j_{l-1}+1}^{N_c-1} \sum_{\tau_{B,1} \in} \sum_{h=1}^{N_c-N_s-1} C_{h,1,N_c-N_s-1} \exp \left(- \left(\frac{x}{\bar{\gamma}_{i_{h,1}}} + \frac{y}{\bar{\gamma}_{i_{k,1}}} \right) \right) \\ & \times \left\{ I \left(z_2, \beta', \frac{y}{N_s}, \frac{x}{N_c-N_s}; z_4, \alpha''', 0, \min \left[\frac{y}{N_s}, \frac{y - \frac{l}{N_c-N_s} \cdot x}{(N_s-l)} \right] \right) + I' \left(z_2, \beta', \frac{y}{N_s}, \frac{y - (N_s-l) \cdot z_4}{l}; z_4, \alpha''', 0, \frac{y}{N_s} \right) \right. \\ & \left. \left. - I' \left(z_2, \beta', \frac{y}{N_s}, \frac{y - (N_s-l) \cdot z_4}{l}; z_4, \alpha''', 0, \min \left[\frac{y}{N_s}, \frac{y - \frac{l}{N_c-N_s} \cdot x}{(N_s-l)} \right] \right) \right\} \right] \end{aligned} \quad (37)$$

$$\begin{aligned}
 \text{where } \alpha &= -\left(\sum_{l=N_c-N_s+1}^{N_c} \left(\frac{1}{\bar{\gamma}_{i_{l,1}}}\right) - \frac{(N_s)}{\bar{\gamma}_{i_{k,1}}}\right), \\
 \alpha' &= -\left(\sum_{l=N_c-N_s+1}^{N_c} \left(\frac{1}{\bar{\gamma}_{i_{l,1}}}\right) + \sum_{m=1}^g \frac{1}{\bar{\gamma}_{i_{f_m,1}}} - \frac{(N_s)}{\bar{\gamma}_{i_{k,1}}}\right), \\
 \alpha'' &= -\left(\sum_{l=N_c-N_s+1}^{N_c} \left(\frac{1}{\bar{\gamma}_{i_{l,1}}}\right) - \frac{(N_s-l)}{\bar{\gamma}_{i_{k,1}} - \sum_{q=1}^l \frac{1}{\bar{\gamma}_{i_{q,1}}}}\right), \\
 \alpha''' &= -\left(\sum_{l=N_c-N_s+1}^{N_c} \left(\frac{1}{\bar{\gamma}_{i_{l,1}}}\right) + \sum_{m=1}^g \frac{1}{\bar{\gamma}_{i_{f_m,1}}} - \frac{(N_s-l)}{\bar{\gamma}_{i_{k,1}}} - \sum_{q=1}^l \frac{1}{\bar{\gamma}_{i_{q,1}}}\right), \\
 \beta &= -\left(\sum_{l=1}^{N_c-N_s} \left(\frac{1}{\bar{\gamma}_{i_{l,1}}}\right) - \frac{N_c-N_s}{\bar{\gamma}_{i_{h,1}}}\right), \text{ and} \\
 \beta' &= -\left(\sum_{l=1}^{N_c-N_s} \left(\frac{1}{\bar{\gamma}_{i_{l,1}}}\right) + \sum_{q=1}^l \frac{1}{\bar{\gamma}_{i_{q,1}}} - \frac{N_c-N_s}{\bar{\gamma}_{i_{h,1}}} - \frac{l}{\bar{\gamma}_{i_{k,1}}}\right). \text{ Further,} \\
 \tau_{B,1} &= \tau_1(i, 1, N_c - N_s - 1) = \{i_{1,1}, i_{2,1}, \dots, i_{N_c-N_s-1,1}\}, \\
 \tau_{B,2} &= \tau_1(i, N_c - N_s, N_c - N_s) = \{i_{N_c-N_s,1}\}, \\
 \tau_{B,3} &= \tau_1(i, N_c - N_s + 1, N_c - 1) = \{i_{N_c-N_s+1,1}, i_{N_c-N_s+2,1}, \dots, i_{N_c-1,1}\}, \text{ and} \\
 \tau_{B,4} &= \tau_1(i, N_c, N_1) = \{i_{N_c,1}, i_{N_c+1,1}, \dots, i_{N_1,1}\}.
 \end{aligned}$$

CDF of the N_c/N_1 -GSC output SNR over i.n.d. Rayleigh fading, $F_{Y+W_1}(x)$

In this case, by replacing N_s and z_1 in [16], Eq (42)] with N_c and $z-z_2$, respectively, and then by applying Eq (25), we can obtain the following double-integral form for Eq (11) as

$$\begin{aligned}
 \int_0^x \int_0^{\frac{x}{\bar{\gamma}_c}} f_Z(z - z_2, z_2) dz_2 dz &= \sum_{i_{N_c}=1}^N \frac{1}{\bar{\gamma}_{i_{N_c}}} \sum_{i_{N_c+1}, \dots, i_N}^{1,2,\dots,N} \sum_{\substack{\{i_1, \dots, i_{N_c-1}\} \in \\ i_{N_c+1} \neq \dots \neq i_N \text{ } P_{N_c-1}(I_N - \{i_{N_c}\} - \{i_{N_c+1}, \dots, i_N\}) \\ \vdots \\ i_N \neq i_{N_c+1}}} \\
 \prod_{q=1}^{N_c} C_{q,1,N_c-1} &\left[-\int_0^x \int_0^{\frac{x}{\bar{\gamma}_c}} \exp\left(-\frac{z}{\bar{\gamma}_{i_q}} - \left(\sum_{l=1}^{N_c} \frac{1}{\bar{\gamma}_{i_l}} - \frac{N_c}{\bar{\gamma}_{i_q}}\right) z_2\right) dz_2 dz \right. \\
 \{i_1, \dots, i_{N_c-1}\} & \\
 - \prod_{k'=1}^{N-N_c} (-1)^{k'} \sum_{j_1=j_0+N_c+1}^{N-k'+1} \dots \sum_{j_{k'}=j_{k'-1}+1}^N \int_0^x \int_0^{\frac{x}{\bar{\gamma}_c}} &\exp\left(-\frac{z}{\bar{\gamma}_{i_q}} - \left(\sum_{l=1}^{N_c} \frac{1}{\bar{\gamma}_{i_l}} + \sum_{m=1}^{k'} \frac{1}{\bar{\gamma}_{i_{j_m}}} - \frac{N_c}{\bar{\gamma}_{i_q}}\right) z_2\right) dz_2 dz \left. \right].
 \end{aligned} \tag{38}$$

In Eq (38), the first and the second double-integral terms can be expressed in closed form by directly applying Eq (35) as

$$\begin{aligned} & \int_0^x \int_0^{\frac{x}{N_c}} \exp\left(-\frac{z}{\bar{\gamma}_{i_q}}\right) \exp\left(-\left(\sum_{l=1}^{N_c} \frac{1}{\bar{\gamma}_{i_l}} - \frac{N_c}{\bar{\gamma}_{i_q}}\right) z_2\right) dz_2 dz \\ &= I'\left(z_2, -\left(\sum_{l=1}^{N_c} \frac{1}{\bar{\gamma}_{i_l}} - \frac{N_c}{\bar{\gamma}_{i_q}}\right), 0, \frac{z}{N_c}; z, -\frac{z}{\bar{\gamma}_{i_q}}, 0, x\right) \end{aligned} \quad (39)$$

and

$$\begin{aligned} & \int_0^x \int_0^{\frac{x}{N_c}} \exp\left(-\frac{z}{\bar{\gamma}_{i_q}}\right) \exp\left(-\left(\sum_{l=1}^{N_c} \frac{1}{\bar{\gamma}_{i_l}} + \sum_{m=1}^{k'} \frac{1}{\bar{\gamma}_{i_{j_m}}} - \frac{N_c}{\bar{\gamma}_{i_q}}\right) z_2\right) dz_2 dz \\ &= I'\left(z_2, -\left(\sum_{l=1}^{N_c} \frac{1}{\bar{\gamma}_{i_l}} + \sum_{m=1}^{k'} \frac{1}{\bar{\gamma}_{i_{j_m}}} - \frac{N_c}{\bar{\gamma}_{i_q}}\right), 0, \frac{z}{N_c}; z, -\frac{z}{\bar{\gamma}_{i_q}}, 0, x\right). \end{aligned} \quad (40)$$

Substituting Eqs (39) and (40) in (38), we can obtain the target CDF in closed form as

$$\begin{aligned} F_{Y+W_1}(x) &= \sum_{i_{N_c,1}=1}^{N_1} \frac{1}{\bar{\gamma}_{i_{N_c,1}}} \sum_{i_{N_c+1,1}, \dots, i_{N_1,1}}^{1,2,\dots,N_1} \sum_{\tau_{A,1} \in} \prod_{q=1}^{N_c} C_{q,1,N_c-1} \\ &\quad i_{N_c+1,1} \neq \dots \neq i_{N_1,1} P_{N_c-1}(I_{N_1} - \tau_{A,2} - \tau_{A,3}) \tau_{A,1} \\ &\quad \vdots \\ &\quad i_{N_1,1} \neq i_{N_c+1,1} \\ &\times \left[-I'\left(z_2, -\left(\sum_{l=1}^{N_c} \frac{1}{\bar{\gamma}_{i_l}} - \frac{N_c}{\bar{\gamma}_{i_q}}\right), 0, \frac{z}{N_c}; z, -\frac{z}{\bar{\gamma}_{i_q}}, 0, x\right) \right. \\ &\quad \left. - \prod_{k'=1}^{N_1-N_c} (-1)^{k'} \sum_{j_1=j_0+N_c+1}^{N_1-k'+1} \dots \sum_{j_{k'}=j_{k'-1}+1}^{N_1} I'\left(z_2, -\left(\sum_{l=1}^{N_c} \frac{1}{\bar{\gamma}_{i_l}} + \sum_{m=1}^{k'} \frac{1}{\bar{\gamma}_{i_{j_m}}} - \frac{N_c}{\bar{\gamma}_{i_q}}\right), 0, \frac{z}{N_c}; z, -\frac{z}{\bar{\gamma}_{i_q}}, 0, x\right) \right] \\ &\quad \tau_{A,3} \end{aligned} \quad (41)$$

where $j_0 = 0$ and

$$C_{l,n_1,n_2} = \frac{1}{\prod_{l=n_1}^{n_2} (-\bar{\gamma}_{i_{l,1}}) F'_{l,n_1,n_2}\left(\frac{1}{\bar{\gamma}_{i_{l,1}}}\right)} \quad (42)$$

where

$$\begin{aligned} F'_{l,n_1,n_2}(x) &= \left[\sum_{l=1}^{n_2-n_1} (n_2 - n_1 - l + 1) x^{n_2-n_1-l} (-1)^l \sum_{j_1=j_0+n_1}^{n_2-l+1} \dots \sum_{j_l=j_{l-1}+1}^{n_2} \prod_{m=1}^l \frac{1}{\bar{\gamma}_{i_{j_m,1}}} \right] \\ &\quad + (n_2 - n_1 + 1) x^{n_2-n_1}. \end{aligned} \quad (43)$$

Further, $\tau_{A,1} = \tau_1(i, 1, N_c - 1) = \{i_{1,1}, i_{2,1}, \dots, i_{N_c-1,1}\}$, $\tau_{A,2} = \tau_1(i, N_c, N_c) = \{i_{N_c,1}\}$, and $\tau_{A,3} = \tau_1(i, N_c + 1, N_1) = \{i_{N_c+1,1}, i_{N_c+2,1}, \dots, i_{N_1,1}\}$.

CDF of the sums of N_s strongest paths from each target BS over i.n.d. Rayleigh fading, $F_{W_n}(x)$

In this case, we can easily obtain the closed-form expression of Eq (17) by directly adopting the derived result in Eq (11), specifically replacing N_c with N_s , as

$$F_{W_n}(x) = \sum_{i_{N_s,n}=1}^{N_n} \frac{1}{\bar{\gamma}_{i_{N_s,n}}} \sum_{\substack{1,2,\dots,N_n \\ i_{N_s+1,n}, \dots, i_{N_n,n} \\ i_{N_s+1,n} \neq \dots \neq i_{N_n,n} \\ \vdots \\ i_{N_n,n} \neq i_{N_s+1,n}}} \sum_{\tau_{C,1} \in \tau_{C,2} = \tau_{C,3}} \prod_{q=1}^{N_s} C_{q,1,N_s-1} \\ \times \left[-I' \left(z_2, - \left(\sum_{l=1}^{N_s} \frac{1}{\bar{\gamma}_{i_l}} - \frac{N_s}{\bar{\gamma}_{i_q}} \right), 0, \frac{z}{N_s}; z, -\frac{z}{\bar{\gamma}_{i_q}}, 0, x \right) \right. \\ \left. - \prod_{k'=1}^{N_n-N_s} (-1)^{k'} \sum_{j_1=j_0+N_s+1}^{N_n-k'+1} \dots \sum_{j_{k'}=j_{k'-1}+1}^{N_n} I' \left(z_2, - \left(\sum_{l=1}^{N_s} \frac{1}{\bar{\gamma}_{i_l}} + \sum_{m=1}^{k'} \frac{1}{\bar{\gamma}_{i_{j_m}}} - \frac{N_s}{\bar{\gamma}_{i_q}} \right), 0, \frac{z}{N_s}; z, -\frac{z}{\bar{\gamma}_{i_q}}, 0, x \right) \right], \quad (44)$$

where $\tau_{C,1} = \tau_n(i, 1, N_s - 1) = \{i_{1,n}, i_{2,n}, \dots, i_{N_s-1,n}\}$, $\tau_{C,2} = \tau_n(i, N_s, N_s) = \{i_{N_s,n}\}$, and $\tau_{C,3} = \tau_n(i, N_s + 1, N_1) = \{i_{N_s+1,n}, i_{N_s+2,n}, \dots, i_{N_1,n}\}$.

Note that while [2] provides the non-closed-form expressions even over i.i.d. fading assumptions since the final results involve finite integrations, in this report we provide all three required key statistics in Eqs (37), (41) and (44) in closed form to accurately investigating the performance measures. With these derived joint statistics, the outage probability as well as other performance measures mentioned in the 'System Models and Performance Measures' section, can be easily calculated with standard mathematical softwares such as Mathematica.

Simulation results and conclusions

In this work, we studied the assessment tool for the finger replacement scheme proposed in [2] over i.n.d. fading conditions by providing the general comprehensive mathematical framework. Specifically, we provided the closed-form expressions for the required key statistics of i. n.d. ordered exponential RVs by applying the unified framework proposed in [15, 16]. Also, the general comprehensive frameworks for the outage performance based on these statistical results are shown.

In Fig 2, we assess the effect of non-identically distributed paths on the outage performance of the replacement schemes. The comparative analysis results are shown in Table 1. More specifically, instead of the uniform PDP, we now consider an exponentially decaying PDP. Experimental measurements indicate that the radio channel is characterized by an exponentially decaying multipath intensity profile (MIP) for indoor office buildings [19] as well as urban [20] and suburban areas [21]. We assume that the channel has an exponential MIP, for which $\bar{\gamma}_i = \bar{\gamma} \cdot \exp(-\delta(i-1))$, ($1 \leq i \leq N_n$, $1 \leq n \leq L$) where $\bar{\gamma}_i$ is the average SNR of the i -th path out of the total available resolvable paths from each BS, $\bar{\gamma}$ is the strongest average SNR (or the average SNR of the first path), and δ is the average fading power decay factor where $\delta = 0$ means identically distributed paths. These results show that the effect of path unbalance induces non-negligible performance degradation compared with the results for i.i.d. fading

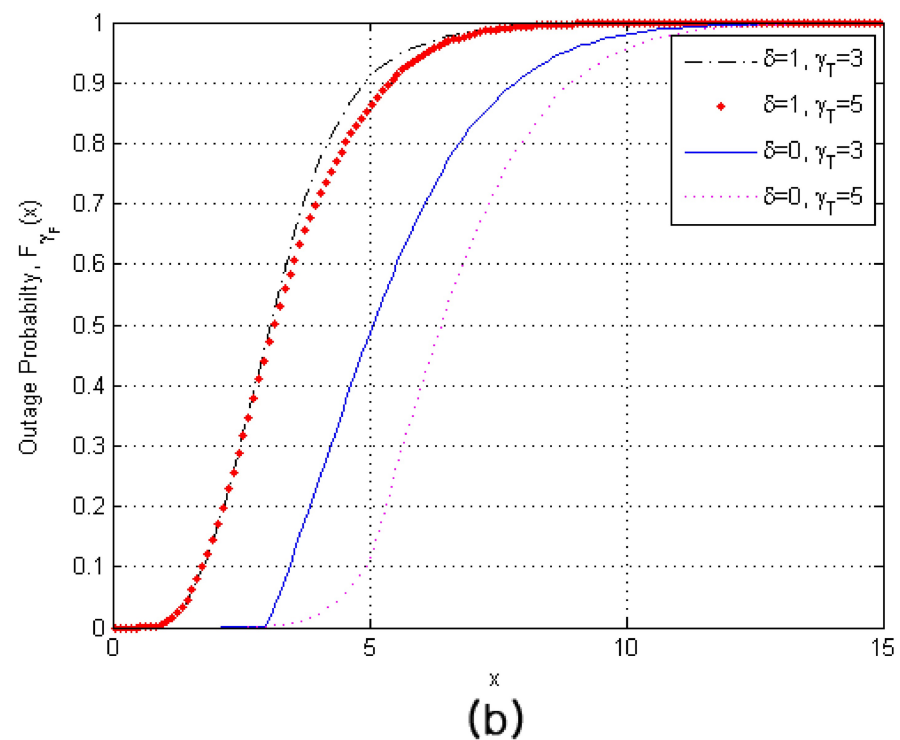
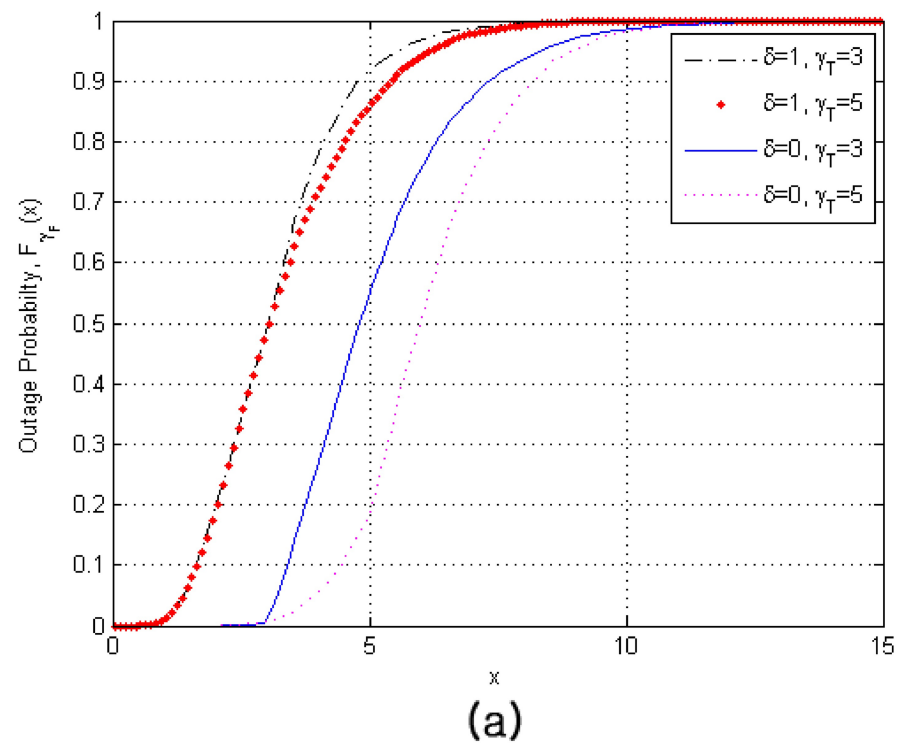


Fig 2. Outage probability of finger replacement schemes for RAKE receivers over i.n.d. Rayleigh fading channels when $L = 4$, $N_1 = \dots = N_4 = 5$, and $N_c = 3$. (a) $N_s = 1$ (b) $N_s = 2$.

<https://doi.org/10.1371/journal.pone.0179126.g002>

Table 1. Outage probability ($F_{\gamma_A}(x)$) when $L = 4$, $N_1 = \dots = N_4 = 5$, and $N_c = 3$.

x	(a) $N_s = 1$				(b) $N_s = 2$			
	$\delta = 1$ $\gamma_T = 3$	$\delta = 1$ $\gamma_T = 5$	$\delta = 0$ $\gamma_T = 3$	$\delta = 0$ $\gamma_T = 5$	$\delta = 1$ $\gamma_T = 3$	$\delta = 1$ $\gamma_T = 5$	$\delta = 0$ $\gamma_T = 3$	$\delta = 0$ $\gamma_T = 5$
0	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
1	0.014	0.014	0.000	0.000	0.008	0.009	0.000	0.000
2	0.211	0.201	0.000	0.000	0.171	0.170	0.000	0.000
3	0.508	0.499	0.018	0.009	0.491	0.471	0.013	0.003
4	0.790	0.725	0.282	0.061	0.778	0.718	0.253	0.025
5	0.920	0.863	0.562	0.201	0.914	0.864	0.495	0.122
6	0.970	0.944	0.762	0.518	0.964	0.946	0.697	0.411
7	0.988	0.979	0.878	0.760	0.987	0.982	0.835	0.651
8	0.995	0.992	0.938	0.888	0.996	0.993	0.915	0.811
9	0.998	0.997	0.972	0.954	0.998	0.997	0.959	0.907
10	1.000	0.999	0.986	0.982	0.999	0.999	0.980	0.959
11	1.000	1.000	0.993	0.994	1.000	1.000	0.991	0.982
12	1.000	1.000	0.997	0.997	1.000	1.000	0.996	0.993
13	1.000	1.000	0.999	0.999	1.000	1.000	0.998	0.997
14	1.000	1.000	1.000	1.000	1.000	1.000	0.999	0.999
15	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000

<https://doi.org/10.1371/journal.pone.0179126.t001>

scenario. This effect must be taken into account for the accurate prediction of the performance over i.n.d. fading environments and, with our analytical results, we believe that our results make it available more easily. In addition, we can also observe from this figure that the bigger the size of the comparison block, N_s , the better the performance, especially when $\delta = 0$ while the difference of the outage performance is negligible when $\delta = 1$. This is because there is a higher chance to have a better group among all W_n over an i.i.d. assumption but in case of i.n.d. environment, the difference in channel quality among all W_n is relatively small. Note that W_1 is the sum of the N_s smallest paths among the N_c currently used paths from the serving BS while $W_n (2 \leq n \leq L)$ are the sums of the N_s strongest paths from each target BS.

Fig 3 shows the outage probability for various values of δ and N_c . The comparative analysis results are shown in Table 2. As expected, the increase of the number of combined paths, N_c , will provide the better performance. However, as δ increases, the performance difference is reduced because of the same reason mentioned in Fig 2 such that the increment of the combined signal strength can be relatively smaller compared with that for small δ .

In summary, we clearly see from these results the importance of the non-identical fading channel effect on the performance. Based on above results, our analytical results can help the system designer to predict the performance and to design the system (e.g., the number of combined paths and so on) accordingly in order to reduce the effect of decay exponents before applying it to the practical environment.

Our results are much simpler than the multiple-fold integral form based on the conventional MGF approach. Moreover, while it is almost impossible to estimate the required performance measure accurately especially for a large number of resolvable paths even with the conventional mathematical tools, our derived closed-form results, even if they look somewhat messy, make the probabilistic analysis available numerically with the conventional mathematical tools. Note that the slightly modified mathematical analytical framework suitable for the derived joint statistical results can be configured to be directly applicable to other various fading scenarios while the conditional PDF based approach and related results in the previous

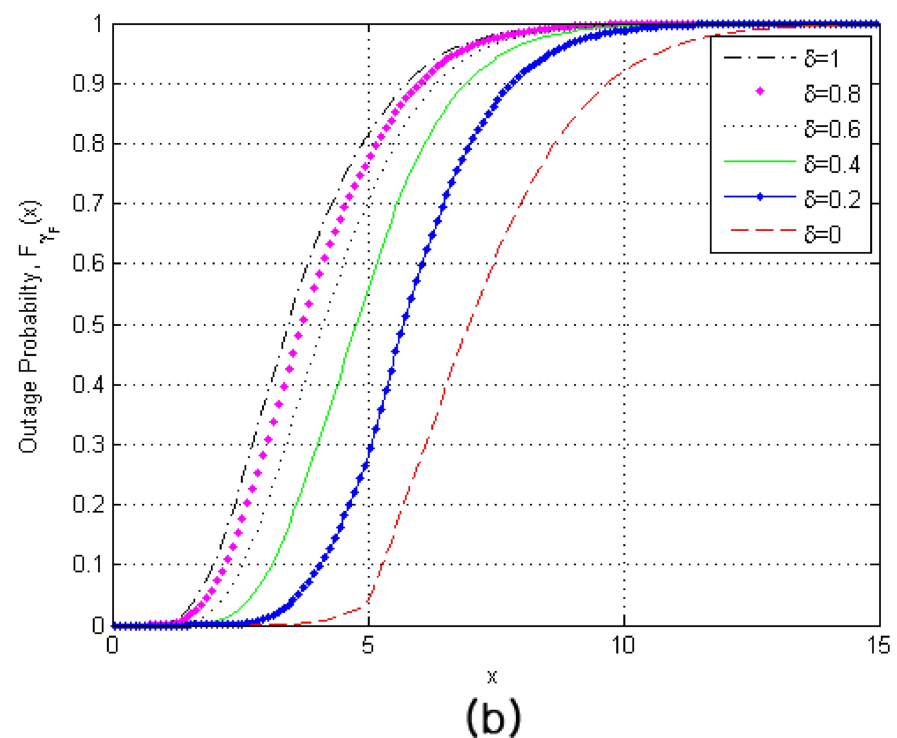
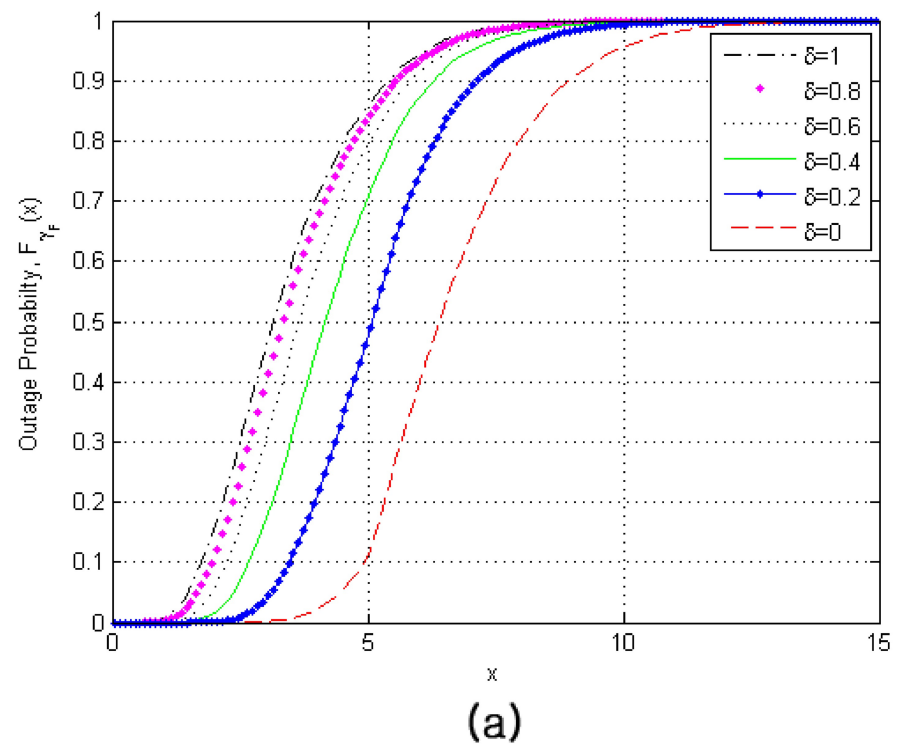


Fig 3. Outage probability of finger replacement schemes for RAKE receivers in the soft handover region for various values of δ when $L = 4$, $N_1 = \dots = N_4 = 5$, $N_s = 2$, and $\gamma_T = 5$. (a) $N_c = 3$ (b) $N_c = 4$.

<https://doi.org/10.1371/journal.pone.0179126.g003>

Table 2. Outage probability ($F_{\gamma_A}(x)$) for various values of δ when $L = 4$, $N_1 = \dots = N_4 = 5$, $N_s = 2$, and $\gamma_T = 5$.

x	(a) $N_c = 3$						(b) $N_c = 4$					
	$\delta = 1$	$\delta = 0.8$	$\delta = 0.6$	$\delta = 0.4$	$\delta = 0.2$	$\delta = 0$	$\delta = 1$	$\delta = 0.8$	$\delta = 0.6$	$\delta = 0.4$	$\delta = 0.2$	$\delta = 0$
0	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
1	0.009	0.003	0.001	0.000	0.000	0.000	0.004	0.001	0.000	0.000	0.000	0.000
2	0.170	0.123	0.067	0.020	0.001	0.000	0.116	0.074	0.032	0.006	0.000	0.000
3	0.471	0.413	0.320	0.183	0.044	0.003	0.389	0.309	0.209	0.090	0.013	0.000
4	0.718	0.681	0.607	0.468	0.221	0.025	0.652	0.584	0.480	0.309	0.099	0.006
5	0.864	0.843	0.801	0.718	0.489	0.122	0.821	0.779	0.712	0.568	0.293	0.048
6	0.946	0.936	0.918	0.877	0.754	0.411	0.924	0.903	0.870	0.791	0.597	0.282
7	0.982	0.977	0.969	0.952	0.892	0.651	0.971	0.963	0.947	0.911	0.807	0.516
8	0.993	0.991	0.988	0.981	0.957	0.811	0.988	0.985	0.979	0.966	0.919	0.713
9	0.997	0.997	0.996	0.993	0.984	0.907	0.995	0.994	0.992	0.987	0.969	0.844
10	0.999	0.999	0.998	0.997	0.994	0.959	0.998	0.998	0.998	0.995	0.988	0.923
11	1.000	1.000	0.999	0.999	0.998	0.982	0.999	0.999	0.999	0.998	0.996	0.964
12	1.000	1.000	1.000	1.000	0.999	0.993	1.000	1.000	1.000	0.999	0.999	0.983
13	1.000	1.000	1.000	1.000	1.000	0.997	1.000	1.000	1.000	1.000	0.999	0.992
14	1.000	1.000	1.000	1.000	1.000	0.999	1.000	1.000	1.000	1.000	1.000	0.997
15	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000

<https://doi.org/10.1371/journal.pone.0179126.t002>

work was limited only to i.i.d. Rayleigh fading assumptions. Finally, as investigated in [22, 23], with the approximation technique such as a central limit theorem, our results can be further generalized to the case of dependent RVs. However, since a thorough discussion on this is not suitable for this work and beyond the scope of this paper, we will consider this approximation in an analytical sense as a future work.

Acknowledgments

This paper was supported by Wonkwang University in 2017.

Author Contributions

Formal analysis: SSN.

Methodology: MSA.

Validation: SC.

Writing – review & editing: YCK.

References

1. Choi S, Alouini M.-S, Qaraqe K. A, Yang H.-C. Finger replacement method for RAKE receivers in the soft handover region. IEEE Trans. Wireless Commun. 2008 Apr; 7(4): 1152–1156. <https://doi.org/10.1109/TWC.2008.060902>
2. Choi S, Alouini M.-S, Qaraqe K. A, Yang H.-C. Finger replacement schemes for RAKE receivers in the soft handover region with multiple base stations. IEEE Trans. Veh. Technol. 2008 July; 57(4): 2114–2122. <https://doi.org/10.1109/TVT.2007.912162>
3. Choi S, Alouini M.-S, Qaraqe K. A, Yang H.-C. Soft handover overhead reduction by RAKE reception with finger reassignment. IEEE Trans. Commun. 2008 Feb; 56(2): 213–221. <https://doi.org/10.1109/TCOMM.2008.060340>

4. Choi S, Alouini M.-S, Qaraqe K. A, Yang H.-C. Finger assignment schemes for RAKE receivers with multiple-way soft handover. *IEEE Trans. Wireless Commun.* 2008 Feb; 7(2): 495–499. <https://doi.org/10.1109/TWC.2008.060814>
5. Yang H.-C. New results on ordered statistics and analysis of minimum-selection generalized selection combining (GSC). *IEEE Trans. Wireless Commun.* 2006 July; 5(7): 1876–1885. <https://doi.org/10.1109/TWC.2006.1673099>
6. Cassioli D. 60 GHz UWB channel measurement and model. *Proc. of IEEE International Conference on Ultra-Wideband, USA* 2012 Sept: 145–149.
7. Rinaldi C, Rendeviski N, Cassioli D. Performance evaluation of UWB signaling at mm waves. *Proc. of IEEE International Conference on Ultra-Wideband, France* 2014 Sept: 379–384.
8. Renzo M. D. Performance evaluation of UWB signaling at mm waves. *IEEE Trans. Wireless Commun.* 2015 Sept; 14(9): 5038–5057.
9. Annamalai A, Deora G.K, Tellambura C. Theoretical diversity improvement in GSC(N,L) receiver with nonidentical fading statistics. *IEEE Trans. Commun.* 2005 June; 53(6): 1027–1035. <https://doi.org/10.1109/TCOMM.2005.849742>
10. Mo Y, Yu D, Song J, Zheng K, Guo Y (2015) A Beacon Transmission Power Control Algorithm Based on Wireless Channel Load Forecasting in VANETs. *PLoS ONE* 10(11): e0142775. <https://doi.org/10.1371/journal.pone.0142775> PMID: 26571042
11. Kumaravel R, Narayanaswamy K (2015) Performance Enhancement of MC-CDMA System through Novel Sensitive Bit Algorithm Aided Turbo Multi User Detection. *PLoS ONE* 10(2): e0115710. <https://doi.org/10.1371/journal.pone.0115710> PMID: 25714917
12. Tham M-L, Chow C-O, Xu Y-H, Ramli N (2016) Two-Level Scheduling for Video Transmission over Downlink OFDMA Networks. *PLoS ONE* 11(2): e0148625. <https://doi.org/10.1371/journal.pone.0148625> PMID: 26906398
13. Alouini M.-S, Simon M. K. An MGF-based performance analysis of generalized selective combining over Rayleigh fading channels. *IEEE Trans. Commun.* 2000 Mar; 48(3): 401–415. <https://doi.org/10.1109/26.837044>
14. Nam S. S, Alouini M.-S, Yang H.-C. A MGF-based unified framework to determine the joint statistics of partial sums of ordered random variables. *IEEE Trans. Inform. Theory* 2010 Nov; 56(8): 5655–5672. <https://doi.org/10.1109/TIT.2010.2070271>
15. Nam S. S, Yang H.-C, Alouini M.-S, Kim D. I. Exact capture probability analysis of GSC receivers over i. n.d. Rayleigh fading channels. *Proc. of IEEE International Symposium on Information Theory and its Applications Turkey* 2013 July: 56–60.
16. Nam S. S, Alouini M.-S, Yang H.-C, Kim D. I. An MGF-based unified framework to determine the joint statistics of partial sums of ordered i.n.d. random variables. *IEEE Trans. Signal Processing* 2014 May; 62(16): 4270–4283. <https://doi.org/10.1109/TSP.2014.2326624>
17. Papoulis A. *Probability, Random Variables, and Stochastic Processes*. NY: McGraw-Hill; 1991.
18. Abramowitz M, Stegun I. A. *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*. NY: Dover Publications; 1970.
19. Hashemi H. Impulse response modeling of indoor radio propagation channels. *IEEE J. Select. Areas Commun.* 1993 Sept; 11(7): 967–978. <https://doi.org/10.1109/49.233210>
20. Erceg V, Michelson D. G, Ghassemzadeh S. S, Greenstein L. J, Rustako J. A. Jr, Gurelain P. B, et al. A model for the multipath delay profile of fixed wireless channels. *IEEE J. Select. Areas Commun.* 1999 Mar; 17(3): 399–410. <https://doi.org/10.1109/49.753726>
21. Turin G. L, Clapp F. D, Johnston T. L, Fine S. B, Lavry D. A statistical model of urban multipath propagation. *IEEE Trans. Veh. Technol.* 1972 Feb; 21(2): 1–9. <https://doi.org/10.1109/T-VT.1972.23492>
22. Shang Y. Central limit theorem for the sum of a random number of dependent random variables. *Asian Journal of Mathematics & Statistics*.
23. Shang Y. A central limit theorem for randomly indexed m-dependent random variables. *Filomat*.