RESEARCH ARTICLE

Optimal and total controllability approach of non-instantaneous Hilfer fractional derivative with integral boundary condition

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Abstract

The focus of this work is on the absolute controllability of Hilfer impulsive non-instantaneous neutral derivative (HINND) with integral boundary condition of any order. Total controllability refers to the system's ability to be controlled during the impulse time. Kuratowski measure and semigroup theory in Banach space yield the results. Furthermore, we talked about optimal controllability in conjunction with appropriate limitations. Our established outcomes are described using an example.

1 Introduction

The concept of differential equations with non-instantaneous impulses(NII) involves many physical processes due to its tremendous applications. Impulse is an action, that starts at an arbitrary fixed point and remains active on a finite time interval is called as NI impulse that occurs in many physical processes like harvesting, vaccination, natural disasters, and shocks subjected to unexpected change in their state. The above situations have to be modeled by impulses [1, 2] if necessary that can not be solved using ordinary differential equations. For some processes, instantaneous impulsive dynamic systems do not support a perfect description, for example, endorsement of insulin of hyperglycemia patients. The change in the above system caused by this medication will remain until the total absorption for a finite time, thanks to the evolutionary process can be modeled with NII. This theory is originated by Hernández [3]. Recently, Vipin Kumar et al. [4–7] derived the controllability results of fractional systems with and without NII for various models. To seek more about NI impulse, track and surf the articles [8–16] and cited references.

On the other hand, the existence and controllability theory extended for both DEs of integer and non-integer order with NII. Fractional calculus is the most appropriate way to evaluate the exact solutions to the given model. The results on Caputo and R-L fractional derivatives were discussed in [17–19]. Theory on HFD was introduced by Hilfer [20] and the results are discussed in [21–24]. One can refer to the monographs [25–29] to know more

about fractional derivatives. In general, controllability enables directing the system from a random initial state to the desired ultimate state. The articles [30–35] discuss the controllability results of Caputo and Hilfer fractional differential system in the nondense domain. Furthermore, the existence and controllability of the Hilfer fractional system with infinite delay were examined in [36, 37]. The exact controllability for Hilfer fractional differential inclusions including nonlocal initial conditions was examined by Du et al. [38]. The approximate controllability results for the Hilfer fractional system were derived by [39, 40]. Recently, a prospective field in control systems is optimal control studied in [41–43]. Ultimately, is more appropriate to evaluate them using an optimization procedure involving fractional differential equations.

The outcome of the existence of HINND of arbitrary order was discussed in [44]. Moreover, results on total controllability fractional neutral non-instantaneous system discussed by [45]. In addition, optimization of the non-instantaneous neutral fractional system is investigated by in [46]. No article was found in the existing literature about the investigation of total controllability using semigroup theory.

We contribute this article to analyze the total controllability & optimal control results for HINND of arbitrary order as:

$$\mathcal{D}_{\mathfrak{e}_k}^{p_1,p_2}\Big[\mathfrak{z}(\mathfrak{t}) - \mathcal{K}(\mathfrak{t},\mathfrak{z}(\mathfrak{t}))\Big] = A\Big[\mathfrak{z}(\mathfrak{t}) - \mathcal{K}(\mathfrak{t},\mathfrak{z}(\mathfrak{t}))\Big] + B\mathfrak{u}(\mathfrak{t}) + \mathfrak{F}\Big(\mathfrak{t},\mathfrak{z}(\mathfrak{t})\Big), \quad \mathfrak{t} \in \bigcup_{k=0}^{N}(\mathfrak{e}_k,\mathfrak{t}_{k+1}]$$

$$\mathfrak{z}(\mathfrak{t}) = \frac{1}{\Gamma(p_1)}\int_{\mathfrak{t}_k}^{\mathfrak{t}} (\mathfrak{t}-\omega)^{p_1-1}\mathscr{I}_k(\omega,\mathfrak{z}(\mathfrak{t}_k^-))d\omega, \quad \mathfrak{t} \in \bigcup_{k=1}^{N}(\mathfrak{t}_k,\mathfrak{e}_k) \qquad (1.1)$$

$$I_{0^+}^{(1-\eta)}\mathfrak{z}(0) = \lambda\int_0^T \mathfrak{z}(\omega)d\omega + c, \quad c \in \mathbb{R}.$$

Here, $\mathcal{Y}$ be a Banach space and $A : \mathcal{D}(A) \subset \mathcal{Y} \to \mathcal{Y}$ is closed together with $\mathcal{D}(A) \in \mathcal{Y}$. $\mathcal{D}_{\mathfrak{e}_k}^{p_1,p_2}$ represents Hilfer derivative of fractional order with $0 < p_1 < 1$, $0 \leq p_2 \leq 1$. Also, $\eta = p_1 + p_2 - p_1 p_2$, $\mathfrak{t} \in \mathcal{I} = [0,T]$, $T > 0$. Here $\mathcal{K} : \mathcal{I} \times \mathcal{Y} \to \mathcal{D}(A) \subset \mathcal{Y}$, $\mathfrak{F} : \mathcal{I} \times \mathcal{Y} \to \mathcal{D}(A) \subset \mathcal{Y}$, $\mathscr{I}_k : [\mathfrak{t}_k,\mathfrak{e}_k] \times \mathcal{Y} \to \mathcal{Y}$ are relevant functions. $\mathfrak{t}_k,\mathfrak{e}_k$ fulfills $0 = \mathfrak{t}_0 = \mathfrak{e}_0 < \mathfrak{t}_1 < \mathfrak{e}_1 < \mathfrak{t}_2 < \ldots < \mathfrak{e}_N < \mathfrak{t}_{N+1} = T$. Moreover, $\mathfrak{z}(\mathfrak{t}_k^-) = \lim_{h \to 0^+}\mathfrak{z}(\mathfrak{t}_k - h)$. $B : U \to \mathcal{Y}$ is a bounded linear operator and $\mathfrak{u}(\cdot) \in L^2[\mathcal{I},U]$. The integral boundary condition $\lambda = +1$ or -1. We briefly orchestrated our objective of this work:

(*i*) By incorporating HFD with semigroup operator theory and LT, we have introduced the integral solution of (1.1).

(*ii*) Kuratowski's measure with κ–set-contraction theory has been supported very much to the total controllability of HINND with C_0 semigroup operator for the first time in the literature.

(*iii*) The results on optimal controllability of HINND had been discussed via Lipschitz continuity.

(*iv*) We have gone through with an illustration that enables our analytical outcomes existence.

2 Key notes

The space of continuous functions is defined by $C(\mathcal{I},\mathcal{Y})$ be a provided $||\mathfrak{z}|| = \sup_{\mathfrak{t}\in\mathcal{I}}||\mathfrak{z}(\mathfrak{t})||$.

$$C_{1-\eta}(\mathcal{I},\mathcal{Y}) = \{\mathfrak{z} : \mathcal{Y} \to \mathcal{Y} \text{ provided } \mathfrak{t}^{1-\eta}\mathfrak{z}(\mathfrak{t}) \in C(\mathcal{I},\mathcal{Y})\}, ||\mathfrak{z}||_{C_{1-\eta}} = \sup_{0\leq\mathfrak{t}\leq T}|\mathfrak{t}^{1-\eta}\mathfrak{z}(\mathfrak{t})|.$$

Let $PC_{1-\eta}((t_k, t_{k+1}], \mathcal{Y})$ defines the space of piecewise functions as

$$PC_{1-\eta}(\mathcal{I}, \mathcal{Y}) = \left\{ \begin{array}{l} (t - t_k)^{1-\eta}\mathfrak{z}(t) \in C_{1-\eta}((t_k, t_{k+1}], \mathbb{R}) \\ \lim_{t \to t_k}(t - t_k)^{1-\eta}\mathfrak{z}(t), \ \ k = 1, 2, \ldots, N, \end{array} \right\}$$

provided

$$||\mathfrak{z}|| = \max\left\{ \sup_{t \in \mathcal{I}} ||t^{1-\eta}\mathfrak{z}(t^+)||, \ \sup_{t \in \mathcal{I}} ||t^{1-\eta}\mathfrak{z}(t^-)|| \right\}.$$

$L(\mathcal{Y})$, characterize the space of all bounded linear operators on $\mathcal{Y}$. A, generates the semigroup $\{S_{p_1,p_2}(t)\}$ where $t \geq 0$ with $\sup||S_{p_1,p_2}(t)||_{L(\mathcal{Y})} = \mathcal{M}$. Define a convex, bounded and closed set $\ell = \{\mathfrak{z} \in PC_{1-\eta}(\mathcal{I}, \mathcal{Y}), \ ||\mathfrak{z}(t)|| < \mathfrak{r}, \ t \in \mathcal{I}, \ \mathfrak{r} > 0\}$ in $PC_{1-\eta}(\mathcal{I}, \mathcal{Y})$.

Definition 2.1 [20]. *For $n - 1 < p_1 < n$, $n \in N$ and $p_2 \in (0, 1]$, HFD is defined by:*

$$\mathcal{D}_{0^+}^{p_1,p_2} y(t) = \mathscr{I}_{0^+}^{p_1(n-p_2)} \frac{d}{dt} \mathscr{I}_{0^+}^{(1-p_1)(n-p_2)} y(t) = \mathscr{I}_{0^+}^{p_1(n-p_2)} \mathcal{D}_{0^+}^{p_2+p_1 n - p_2 p_1} y(t),$$

where $\mathcal{D}_{0^+}^{p_2+p_1 n - p_2 p_1}$ and $\mathscr{I}_{0^+}^{p_1(n-p_2)}$ are R-L derivative and integral respectively.

Definition 2.2 [8, 44, 47]. *The Kuratowski noncompact measure $\ell(\cdot)$ characterized as:*

$\ell(\hbar) := \inf\{\rho > 0 : \hbar = \bigcup_{i=1}^{m} \text{ with } diam(\hbar_i) \leq \rho\}$, *where $\hbar$ is a bounded set on $\mathcal{Y}$.*

Lemma 2.3. *(see [8, 44, 47]) For $\hbar_1, \hbar_2 \subset \mathcal{Y}$, the Kuratowski noncompact measure meets:*

1. $\ell(\hbar) = \ell(\overline{\hbar}) = \ell(conv\hbar)$;

2. $\ell(\hbar) = 0$ *iff $\overline{\hbar}$ is compact;*

3. *for given $\lambda \in R$, $\ell(\lambda\hbar) \leq |\lambda|\ell(\hbar)$;*

4. $\hbar_1 \subset \hbar_2$ *implies $\ell(\hbar_1) \leq \ell(\hbar_2)$;*

5. $\ell(\hbar_1 \cup \hbar_2) = \max\{\ell(\hbar_1), \ell(\hbar_2)\}$;

6. $\ell(\hbar_1 + \hbar_2) \leq \ell(\hbar_1) + \ell(\hbar_2)$, *where $\hbar_1 + \hbar_2 = \{\mathfrak{z} \mid \mathfrak{z} = \mathfrak{z}_1 + \mathfrak{z}_2; \ \mathfrak{z}_1 \in \hbar_1, \ \mathfrak{z}_2 \in \hbar_2\}$;*

7. *The Lipschitz function $\Re : \mathcal{D}(\Re) \subset \mathcal{Y} \to \mathcal{Y}$ and the subset $W \subset \mathcal{D}(\Re)$, $\ell(\Re(W)) \leq \kappa \ell(W)$ is bounded.*

Let $\mathcal{D} \subset C_{1-\eta}(\mathcal{I}, \mathcal{Y})$ and $t \in \mathcal{I}$, $\mathcal{D}(t) = \{\mathfrak{z}(t) \mid \mathfrak{z} \in \mathcal{D}\}$ and $\ell(\mathcal{D}(t)) \leq \ell C_{1-\eta}(\mathcal{D})$.

Lemma 2.4. *(see [8, 44, 47]) Let $\mathcal{D} \subset C_{1-\eta}([c_1, c_2], \mathcal{Y})$ be bounded and equicontinuous such that*

$$\ell C_{1-\eta}(\mathcal{D}) = \max_{t \in [c_1, c_2]} \ell(\mathcal{D}(t)),$$

and $\ell(\mathcal{D}(t))$ is continuous on $[c_1, c_2]$.

Lemma 2.5. *(see [8, 44, 47]) Assume that $\overline{\mathcal{Y}} \subset \mathcal{Y}$ is bounded and for some $\mathcal{D}_0 \subset \mathcal{D}$, the countable set meets $\ell(\mathcal{D}) \leq 2\ell(\mathcal{D}_0)$.*

Lemma 2.6. *(see [8, 44, 47]) Let $\mathcal{D} = \{\mathfrak{z}_n\} \subset PC_{1-\eta}([c_1, c_2], \mathcal{Y})$ where $-\infty < c_1 < c_2 < \infty$. Hence $\ell(\mathcal{D}(t))$ on $[c_1, c_2]$ such that:*

$$\ell\left(\left\{\int_{c_1}^{c_2} \mathfrak{z}_n(t)dt\right\}\right) \leq 2\int_{c_1}^{c_2} \ell(\mathcal{D}(t))dt, \ \ n \in \mathbb{N}.$$

Lemma 2.7. (*see* [21, 22, 44]) *The system* (1.1) *becomes*:

$$
\begin{aligned}
\mathfrak{z}(\mathfrak{t}) \;=\;& \frac{\mathfrak{t}^{\eta-1}}{\Gamma(\eta)}\left[\lambda\int_0^T \mathfrak{z}(\omega)d\omega + c - \mathcal{K}(0,\mathfrak{z}(0))\right] + \mathcal{K}(\mathfrak{t},\mathfrak{z}(\mathfrak{t})) \\
&+ \frac{1}{\Gamma(p_1)}\int_0^{\mathfrak{t}} (\mathfrak{t}-\omega)^{p_1-1}\Big[A[\mathfrak{z}(\omega)-\mathcal{K}(\omega,\mathfrak{z}(\omega))] + Bu(\omega) + \mathfrak{F}(\omega,\mathfrak{z}(\omega))\Big]d\omega,\ \ \mathfrak{t}\in(0,\mathfrak{t}_1],
\end{aligned}
$$

$$
\mathfrak{z}(\mathfrak{t}) \;=\; \frac{1}{\Gamma(p_1)}\int_{\mathfrak{t}_k}^{\mathfrak{t}} (\mathfrak{t}-\omega)^{p_1-1}\mathscr{I}_k(\omega,\mathfrak{z}(\mathfrak{t}_k^-))d\omega, \qquad \mathfrak{t}\in(\mathfrak{t}_k,\mathfrak{e}_k],
$$

$$
\begin{aligned}
\mathfrak{z}(\mathfrak{t}) \;=\;& \frac{1}{\Gamma(p_1)}\int_{\mathfrak{t}_k}^{\mathfrak{e}_k} (\mathfrak{t}-\omega)^{p_1-1}\mathscr{I}_k(\omega,\mathfrak{z}(\mathfrak{t}_k^-))d\omega - \mathcal{K}(\mathfrak{e}_k,\mathfrak{z}(\mathfrak{e}_k)) + \mathcal{K}(\mathfrak{t},\mathfrak{z}(\mathfrak{t})) \\
&+ \frac{1}{\Gamma(p_1)}\int_{\mathfrak{e}_k}^{\mathfrak{t}} (\mathfrak{t}-\omega)^{p_1-1}\Big[A\Big[\mathfrak{z}(\omega)-\mathcal{K}(\omega,\mathfrak{z}(\omega))\Big] \\
&+ Bu(\omega) + \mathfrak{F}(\omega,\mathfrak{z}(\omega))\Big]d\omega, \qquad \mathfrak{t}\in(\mathfrak{e}_k,\mathfrak{t}_{k+1}].
\end{aligned}
$$

Definition 2.8. (*see* [21, 22, 44]) *A function* $\mathfrak{z}\in PC_{1-\eta}(\mathcal{I},\mathcal{Y})$ *is a solution of* (1.1), *if*

$(i)\mathfrak{z}(\mathfrak{t}) = \frac{1}{\Gamma(p_1)}\int_{\mathfrak{t}_k}^{\mathfrak{t}} (\mathfrak{t}-\omega)^{p_1-1}\mathscr{I}_k(\omega,\mathfrak{z}(\mathfrak{t}_k^-))d\omega,\ \ \mathfrak{t}\in(\mathfrak{t}_k,\mathfrak{e}_k],\ k=1,2,...,N$

$(ii)I_0^{1-\eta}\mathfrak{z}(0) = \lambda\int_0^T \mathfrak{z}(\omega)d\omega + c,$

together with

$$
\begin{aligned}
\mathfrak{z}(\mathfrak{t}) \;=\;& S_{p_1,p_2}(\mathfrak{t})\left[\lambda\int_0^T \mathfrak{z}(\omega)d\omega + c - \mathcal{K}(0,\mathfrak{z}(0))\right] + \mathcal{K}(\mathfrak{t},\mathfrak{z}(\mathfrak{t})) \\
&+ \int_0^{\mathfrak{t}} K_{p_1}(\mathfrak{t}-\omega)\Big[Bu(\omega) + \mathfrak{F}(\omega,\mathfrak{z}(\omega))\Big]d\omega, \quad \mathfrak{t}\in(0,\mathfrak{t}_1],
\end{aligned} \tag{2.1}
$$

$$
\begin{aligned}
\mathfrak{z}(\mathfrak{t}) \;=\;& S_{p_1,p_2}(\mathfrak{t}-\mathfrak{e}_k)\left[\frac{1}{\Gamma(p_1)}\int_{\mathfrak{t}_k}^{\mathfrak{e}_k} (\mathfrak{e}_k-\omega)^{p_1-1}\mathscr{I}_k(\omega,\mathfrak{z}(\mathfrak{t}_k^-))d\omega - \mathcal{K}(\mathfrak{e}_k,\mathfrak{z}(\mathfrak{e}_k))\right] + \mathcal{K}(\mathfrak{t},\mathfrak{z}(\mathfrak{t})) \\
&+ \int_{\mathfrak{e}_k}^{\mathfrak{t}} K_{p_1}(\mathfrak{t}-\omega)\Big[Bu(\omega) + \mathfrak{F}(\omega,\mathfrak{z}(\omega))\Big]d\omega,\ \mathfrak{t}\in(\mathfrak{e}_k,\mathfrak{t}_{k+1}],\ k=1,2,...,N.
\end{aligned} \tag{2.2}
$$

$$
T_{p_1}(\mathfrak{t}) = p_1\int_0^{\infty} v\psi_y(v)\Re(\mathfrak{t}^{p_1}v)dv, \quad K_{p_1}(\mathfrak{t}) = \mathfrak{t}^{p_1-1}T_{p_1}(\mathfrak{t}), \quad S_{p_1,p_2}(\mathfrak{t}) = \mathscr{I}_{0^+}^{p_2(1-p_1)}K_{p_1}(\mathfrak{t}).
$$

$$
W_y(v) \;=\; \frac{1}{\pi}\sum_{m=1}^{\infty}(-1)^{m-1}v^{-my-1}\frac{\Gamma(my+1)}{m!}\sin(m\pi\mathfrak{z}),\ \ v\in(0,\infty),
$$

$$
\psi_y(v) \;=\; \frac{1}{\mathfrak{z}}v^{\left(-1-\frac{1}{\mathfrak{z}}\right)}W_y(v^{-\frac{1}{\mathfrak{z}}}) \geq 0.
$$

Lemma 2.9. (*see* [8, 44, 47]) *If a family* $\{S_{p_1}(\mathfrak{t}),\ \mathfrak{t}\geq 0\}\subset \mathcal{B}(\mathcal{Y})$ *satisfies*

(i) *for all* $\mathfrak{z}\in D(A), S_{p_1}(\mathfrak{t})\mathfrak{z} = \mathfrak{z} + I_{0^+}^{p_1}S_{p_1}(\mathfrak{t})Ay,\ \mathfrak{t}\geq 0;$

$(ii)S_{p_1}(\mathfrak{t})$ *is strongly continuous on* $\mathbb{R}_+, S_{p_1}(0) = I;$

$(iii) AS_{p_1}(\mathfrak{t})\mathfrak{z} = S_{p_1}A(\mathfrak{z})$ *for each,* $\mathfrak{z} \in D(A), \ \mathfrak{t} \geq 0, D \subset \mathcal{Y}$.

Then, it is said to be p_1*–times resolvent generator by* A.

Definition 2.10. *A system is defined as totally controllable on* $\mathcal{I}$, *if for* $k = 1, 2, \ldots, N$, *it is controllable on* $(0, \mathfrak{t}_1], (\mathfrak{e}_k, \mathfrak{t}_{k+1}]$ *such that* $\mathfrak{z}(0) = \mathfrak{z}_0$ *and* $\mathfrak{z}(\mathfrak{t}_{k+1}) = \mathfrak{z}_{\mathfrak{t}_{k+1}}$.

For further discussions, we consider the subsequent assumptions as:

(H1) $\mathcal{K} : J_0 \times \mathcal{Y} \to \mathcal{Y}, \ J_0 = \bigcup\limits_{k=0}^{N} (\mathfrak{e}_k, \mathfrak{t}_{k+1}]$ is continuous and for $\mathcal{L}_p, \mu_p > 0$ as

$$||\mathcal{K}(\mathfrak{t}, \hat{\mathfrak{z}}_2) - \mathcal{K}(\mathfrak{t}, \hat{\mathfrak{z}}_1)|| \leq \mathcal{L}_p ||\hat{\mathfrak{z}}_2 - \hat{\mathfrak{z}}_1||, \ \hat{\mathfrak{z}}_2, \ \hat{\mathfrak{z}}_1 \in \mathcal{Y}, \ \mathfrak{t} \in J_0,$$

also $||\mathcal{K}(\mathfrak{t}, \mathfrak{z})|| \leq \mu_p, \ \mathfrak{z} \in \mathcal{Y}, \ \mathfrak{t} \in J_0$;

(H2) for any bounded set $D_1 \subset \mathcal{Y}$, *exist* $\mathcal{L}_{p^*} > 0$, such that

$$\ell(\mathcal{K}(\mathfrak{t}, D_1)) \leq \mathcal{L}_{p^*}\ell(D_1);$$

(H3) Function $\mathfrak{F} : J_0 \times \mathcal{Y} \to \mathcal{Y}$ is continuous with $\mathcal{L}_f > 0$, satisfies

$$||\mathfrak{F}(\mathfrak{t}, \hat{\mathfrak{z}}_1) - \mathfrak{F}(\mathfrak{t}, \hat{\mathfrak{z}}_2)|| \leq \mathcal{L}_f ||\hat{\mathfrak{z}}_1 - \hat{\mathfrak{z}}_2||, \ \hat{\mathfrak{z}}_1, \hat{\mathfrak{z}}_2 \in \mathcal{Y}, \ \mathfrak{t} \in J_0.$$

$$||\mathfrak{F}(\mathfrak{t}, \mathfrak{z})|| \leq \Psi(\mathfrak{t})\wp\left(||\mathfrak{z}||\right) \ \text{and} \ \liminf_{l \to \infty} \frac{\wp(l)}{l} = v < \infty;$$

where $\wp : [0, \infty) \to [0, \infty)$, a non decreasing continuous function, $\psi : \mathcal{I} \to [0, \infty)$, a Lebesgue integrable function and $v > 0$ such that for all $\mathfrak{z} \in \mathcal{Y}$, $\mathfrak{t} \in \mathcal{I}$ and meets $||\mathfrak{z}||_{C_{1-\eta}} \leq l$.

(H4) For $k = 0, 1, \ldots N, \mathcal{L}_k$,

$$\ell(\mathfrak{F}(\mathfrak{t}, \mathfrak{z})) \leq \mathcal{L}_k\ell(D), \ \mathfrak{t} \in J_0 \ \text{with} \ \mathcal{L} = \max_k \mathcal{L}_k,$$

where the subset D of $\mathcal{Y}$ is a countable;

(H5) For $J_k = [\mathfrak{t}_k, \mathfrak{e}_k], \ k = 1, 2, \ldots N, \ \mathscr{I}_k : J_k \times \mathcal{Y} \to \mathcal{Y}$ are continuous functions, for $K_{\mathscr{I}_k} > 0, \ k = 1, 2, \ldots N$, provided for every $\hat{\mathfrak{z}}_1, \hat{\mathfrak{z}}_2 \in \mathcal{Y}$,

$$||\mathscr{I}_k(\mathfrak{t}, \hat{\mathfrak{z}}_1) - \mathscr{I}_k(\mathfrak{t}, \hat{\mathfrak{z}}_2)|| \leq K_{\mathscr{I}_k}||\hat{\mathfrak{z}}_1 - \hat{\mathfrak{z}}_2||, \ \text{for each} \ \mathfrak{t} \in (\mathfrak{t}_k, \mathfrak{e}_k], \ K := \max_{k=0,1,\ldots N} K_{\mathcal{I}_k}.$$

Moreover, $\mathscr{M}_{\mathscr{I}}$, together with $||\mathscr{I}_k(\mathfrak{t}, \mathfrak{z})|| \leq \mathscr{M}_{\mathscr{I}}$;

(H6) $W : L^2(\mathcal{I}, U) \to \mathcal{Y}$ defined by:

$$W\mathfrak{u} = \int_0^a K_{p_1}(a - \omega)B\mathfrak{u}(\omega)d\omega,$$

is invertible. Also, for $\mathscr{M}_b, \mathscr{M}_w \geq 0$, and $||W^{-1}|| \leq \mathscr{M}_w, ||B|| \leq \mathscr{M}_b$.

(H7) Given $\mathcal{L}_{\mathfrak{u}}^* > 0$, for $\ell(\mathfrak{u}(\mathfrak{z}, \mu)) \leq \mathcal{L}_{\mathfrak{u}}^* \mathfrak{t}^{1-\eta}\sigma(\mathfrak{z}, \mu)\ell(\mathfrak{z}(\mu))$, a.e. $\mu \in \mathcal{I}$ and $\sup\limits_{\mathfrak{t} \in \mathcal{I}} \int_0^{\mathfrak{t}} \sigma(\mathfrak{t}, \mu)d\omega = \sigma^* < \infty$.

Conveniently, we assign some notations as follows:

$$\mathcal{C}_2 = \frac{\mathscr{M}_1 \mathscr{M}_b \mathscr{M}_w^m T^{p_1}}{p_1}; \quad k_1 = \max\{\mathscr{M}\lambda T + \mathcal{L}_p, \ \mathscr{M}\left(\frac{K_{\mathscr{I}_k}\mathfrak{t}_{k+1}^{p_1}}{\Gamma(p_1+1)} + \mathcal{L}_p\right) + \mathcal{L}_p, \ \frac{K_{\mathscr{I}_k}\mathfrak{t}_{k+1}^{p_1}}{\Gamma(p_1+1)}\};$$

$$\mathcal{N}_1 = \mathscr{M}\left(\frac{\mathscr{M}_{\mathscr{I}}\mathfrak{t}_{k+1}^{p_1}}{\Gamma(p_1+1)} + \mu_p\right) + \mu_p + \frac{\mathscr{M}_1 \mathfrak{t}_{k+1}^{p_1}}{p_1}\wp(l)||\Psi||_{L[0,\mathfrak{t}_{k+1}]}; \quad \mathcal{C}_1 = \frac{\mathscr{M}_1 \mathscr{M}_b \mathscr{M}_w^1 \mathfrak{t}_1^{p_1}}{p_1};$$

$$\mathscr{M}_1 = \sup||K_{p_1}(\mathfrak{t})||_{L(\mathcal{Y})}; \quad \mathcal{N} = \mathscr{M}\left(\lambda||\mathfrak{z}|| + |c| + \mu_p\right) + \mu_p + \frac{\mathscr{M}_1 \mathfrak{t}_1^{p_1}}{p_1}\wp(l)||\Psi||_{L[0,\mathfrak{t}_1]}.$$

3 Main sequels

Lemma 3.1. *Let $\mathcal{S} \subset \mathcal{Y}$ and $\Re : \mathcal{S} \to \mathcal{Y}$ be called as κ-set-contractive for any bounded set $\aleph$ in $\mathcal{S}$ such that and for $\kappa \in [0, 1)$, as*

$$\ell(\Re(\aleph)) \le \kappa\, \ell(\aleph).$$

Lemma 3.2. *Let $\aleph$ be a convex, bounded and closed subset of $\mathcal{Y}$. If $\Re : \aleph \to \aleph$ is κ-set-contractive. Then $\Re$ has at least one fixed point in $\aleph$.*

Lemma 3.3. *If the assumptions (H1)–(H7) true, hence*

$$
\begin{aligned}
\mathfrak{u}(\mathfrak{t}) \ &= \ W^{-1}\Bigg[\mathfrak{z}_{\mathfrak{t}_1} - S_{p_1,p_2}(\mathfrak{t}_1)\Big[\lambda\int_0^T \mathfrak{z}(\omega)d\omega + c - \mathcal{K}(0,\mathfrak{z}(0))\Big] - \mathcal{K}(\mathfrak{t}_1,\mathfrak{z}(\mathfrak{t}_1)) \\
&\quad - \int_0^{\mathfrak{t}_1} K_{p_1}(\mathfrak{t}_1 - \omega)\mathfrak{F}(\omega,\mathfrak{z}(\omega))d\omega\Bigg], \ \ \mathfrak{t} \in (0, \mathfrak{t}_1],
\end{aligned}
\tag{3.1}
$$

$$
\begin{aligned}
\mathfrak{u}(\mathfrak{t}) \ &= \ W^{-1}\Bigg[\mathfrak{z}_{\mathfrak{t}_{k+1}} - S_{p_1,p_2}(\mathfrak{t}_{k+1} - \mathfrak{e}_k) \\
&\quad (\times)\Big[\frac{1}{\Gamma(p_k)}\int_{\mathfrak{t}_k}^{\mathfrak{e}_k}(\mathfrak{e}_k - \omega)^{p_1-1}\mathscr{I}_k(\omega,\mathfrak{z}(\mathfrak{t}_k^-))d\omega - \mathcal{K}(\mathfrak{e}_k,\mathfrak{z}(\mathfrak{e}_k))\Big] - \mathcal{K}(\mathfrak{t}_{k+1},\mathfrak{z}(\mathfrak{t}_{k+1})) \\
&\quad - \int_{\mathfrak{e}_k}^{\mathfrak{t}_{k+1}} K_{p_1}(\mathfrak{t}_{k+1} - \omega)\mathfrak{F}(\omega,\mathfrak{z}(\omega))d\omega\Bigg], \ \ \mathfrak{t} \in (\mathfrak{e}_k, \mathfrak{t}_{k+1}],
\end{aligned}
\tag{3.2}
$$

drives to $\mathfrak{z}(\mathfrak{t})$ of (1.1) from $\mathfrak{z}(\mathfrak{t}_1) = \mathfrak{z}_{\mathfrak{t}_1}$ and $\mathfrak{z}(\mathfrak{t}_{k+1}) = \mathfrak{z}_{\mathfrak{t}_{k+1}}$, also $||\mathfrak{u}(\mathfrak{t})|| = \mathscr{M}_{\mathfrak{u}}^1, ||\mathfrak{u}(\mathfrak{t})|| = \mathscr{M}_{\mathfrak{u}}^k$ with

$$\mathscr{M}_{\mathfrak{u}}^1 = \mathscr{M}_w^1(||\mathfrak{z}_{\mathfrak{t}_1}|| + \mathcal{N}), \ \ \mathscr{M}_{\mathfrak{u}}^k = \mathscr{M}_w^m(||\mathfrak{z}_{\mathfrak{t}_{k+1}}|| + \mathcal{N}_1), \ \ k = 1, 2, \ldots, N.$$

Proof. For $\mathfrak{t} = \mathfrak{t}_1$,

$$
\begin{aligned}
\mathfrak{z}(\mathfrak{t}_1) \quad &= S_{p_1,p_2}(\mathfrak{t}_1)\Big[\lambda \int_0^T \mathfrak{z}(\omega)d\omega + c - \mathcal{K}(0,\mathfrak{z}(0))\Big] + \mathcal{K}(\mathfrak{t}_1,\mathfrak{z}(\mathfrak{t}_1)) \\
&\quad + \int_0^{\mathfrak{t}_1} K_{p_1}(\mathfrak{t}_1 - \omega)\mathfrak{F}(\omega,\mathfrak{z}(\omega))d\omega \\
&\quad + \int_0^{\mathfrak{t}_1} K_{p_1}(\mathfrak{t}_1 - \tau)W^{-1}\Big[\mathfrak{z}_{\mathfrak{t}_1} - S_{p_1,p_2}(\mathfrak{t}_1)\Big[\lambda \int_0^T \mathfrak{z}(\omega)d\omega + c - \mathcal{K}(0,\mathfrak{z}(0))\Big] \\
&\quad - \mathcal{K}(\mathfrak{t}_1,\mathfrak{z}(\mathfrak{t}_1)) - \int_0^{\mathfrak{t}_1} K_{p_1}(\mathfrak{t}_1 - \omega)\mathfrak{F}(\omega,\mathfrak{z}(\omega))d\omega\Big]d\tau \\
&= S_{p_1,p_2}(\mathfrak{t}_1)\Big[\lambda \int_0^T \mathfrak{z}(\omega)d\omega + c - \mathcal{K}(0,\mathfrak{z}(0))\Big] + \mathcal{K}(\mathfrak{t}_1,\mathfrak{z}(\mathfrak{t}_1)) \\
&\quad + \int_0^{\mathfrak{t}_1} K_{p_1}(\mathfrak{t}_1 - \omega)\mathfrak{F}(\omega,\mathfrak{z}(\omega))d\omega + \mathfrak{z}_{\mathfrak{t}_1} - S_{p_1,p_2}(\mathfrak{t}_1)\Big[\lambda \int_0^T \mathfrak{z}(\omega)d\omega + c - \mathcal{K}(0,\mathfrak{z}(0))\Big] \\
&\quad - \mathcal{K}(\mathfrak{t}_1,\mathfrak{z}(\mathfrak{t}_1)) - \int_0^{\mathfrak{t}_1} K_{p_1}(\mathfrak{t}_1 - \omega)\mathfrak{F}(\omega,\mathfrak{z}(\omega))d\omega \\
&= \mathfrak{z}_{\mathfrak{t}_1},
\end{aligned}
$$

with

$$
\begin{aligned}
||\mathfrak{u}(t)|| \quad &\leq \Big|\Big|W^{-1}\Big[\mathfrak{z}_{\mathfrak{t}_1} - S_{p_1,p_2}(\mathfrak{t}_1)\Big[\lambda \int_0^T \mathfrak{z}(\omega)d\omega + c - \mathcal{K}(0,\mathfrak{z}(0))\Big] - \mathcal{K}(\mathfrak{t}_1,\mathfrak{z}(\mathfrak{t}_1)) \\
&\quad - \int_0^{\mathfrak{t}_1} K_{p_1}(\mathfrak{t}_1 - \omega)\mathfrak{F}(\omega,\mathfrak{z}(\omega))d\omega\Big]\Big|\Big| \\
&\leq \mathcal{M}_w^1\Big(||\mathfrak{z}_{\mathfrak{t}_1}|| + \mathcal{M}\Big|\Big|\lambda \int_0^T \mathfrak{z}(\omega)d\omega + c - \mathcal{K}(0,\mathfrak{z}(0))\Big|\Big| + ||\mathcal{K}(\mathfrak{t}_1,\mathfrak{z}(\mathfrak{t}_1))|| \\
&\quad + \mathcal{M}_1\Big|\Big|\int_0^{\mathfrak{t}_1} \omega^{1-\eta}\Psi(\omega)\wp(||\mathfrak{z}||_{C_{1-\eta}})d\omega\Big|\Big|\Big) \\
&\leq \mathcal{M}_w^1\Big(||\mathfrak{z}_{\mathfrak{t}_1}|| + \mathcal{M}\Big(\lambda T||\mathfrak{z}(\omega)|| + |c| + \mu_p\Big) + \mu_p + \frac{\mathcal{M}_1 \mathfrak{t}_1^{p_1}}{p_1}\wp(l)||\Psi||_{L[0,\mathfrak{t}_1]}\Big) \\
&\leq \mathcal{M}_w^1(||\mathfrak{z}_{\mathfrak{t}_1}|| + \mathcal{N}) \\
&= \mathcal{M}_\mathfrak{u}^1.
\end{aligned}
$$

Also, for $\mathfrak{t} \in (\mathfrak{e}_k, \mathfrak{t}_{k+1}]$ and $\mathfrak{t} = \mathfrak{t}_{k+1}$,

$$
\begin{aligned}
\mathfrak{z}(\mathfrak{t}_{k+1}) \quad &= S_{p_1,p_2}(\mathfrak{t}_{k+1} - \mathfrak{e}_k)\Big[\frac{1}{\Gamma(p_1)}\int_{\mathfrak{t}_k}^{\mathfrak{e}_k}(\mathfrak{e}_k - \omega)^{p_1-1}\mathscr{I}_k(\omega,\mathfrak{z}(\mathfrak{t}_k^-))d\omega - \mathcal{K}(\mathfrak{e}_k,\mathfrak{z}(\mathfrak{e}_k))\Big] \\
&\quad + \mathcal{K}(\mathfrak{t}_{k+1},\mathfrak{z}(\mathfrak{t}_{k+1})) + \int_{\mathfrak{e}_k}^{\mathfrak{t}_{k+1}} K_{p_1}(\mathfrak{t}_{k+1} - \omega)\mathfrak{F}(\omega,\mathfrak{z}(\omega))d\omega \\
&\quad + \int_{\mathfrak{e}_k}^{\mathfrak{t}_{k+1}} K_{p_1}(\mathfrak{t}_{k+1} - \tau)W^{-1}\Big[\mathfrak{z}_{\mathfrak{t}_{k+1}} - S_{p_1,p_2}(\mathfrak{t}_{k+1} - \mathfrak{e}_k) \\
&\quad (\times)\Big[\frac{1}{\Gamma(p_1)}\int_{\mathfrak{t}_k}^{\mathfrak{e}_k}(\mathfrak{e}_k - \omega)^{p_1-1}\mathscr{I}_k(\omega,\mathfrak{z}(\mathfrak{t}_k^-))d\omega - \mathcal{K}(\mathfrak{e}_k,\mathfrak{z}(\mathfrak{e}_k))\Big] \\
&\quad - \mathcal{K}(\mathfrak{t}_{k+1},\mathfrak{z}(\mathfrak{t}_{k+1})) - \int_{\mathfrak{e}_k}^{\mathfrak{t}_{k+1}} K_{p_1}(\mathfrak{t}_{k+1} - \omega)\mathfrak{F}(\omega,\mathfrak{z}(\omega))d\omega\Big]d\tau \\
&= \mathfrak{z}_{\mathfrak{t}_{k+1}},
\end{aligned}
$$

with

$$
\begin{aligned}
||\mathfrak{u}(\mathfrak{t})|| \quad &\le \mathscr{M}_w^m \left(||\mathfrak{z}_{\mathfrak{t}_{k+1}}|| + \mathscr{M} \left(\frac{\mathscr{M}_{\mathscr{I}} \mathfrak{t}_{k+1}^{p_1}}{\Gamma(p_1 + 1)} + \mu_p \right) + \mu_p + \frac{\mathscr{M}_1 \mathfrak{t}_{k+1}^{p_1}}{p_1} \wp(l) ||\Psi||_{L[0,\mathfrak{t}_{k+1}]} \right) \\
&\le \mathscr{M}_w^m (||\mathfrak{z}_{\mathfrak{t}_{k+1}}|| + \mathcal{N}_1) \\
&= \mathscr{M}_{\mathfrak{u}}^m.
\end{aligned}
$$

Theorem 3.4. *The system* (1.1) *is totally controllable on* $\mathcal{I}$*, if it meets the assumptions* (*H*1)–(*H*7) *together with the conditions*

$$
[\mathscr{M}\lambda + 2\mathcal{L}_p^* + \mathscr{M}_{\mathscr{I}}(\mathscr{M}+1) + 4\mathcal{L}_{\mathfrak{u}}^* \sigma^* \mathscr{M}_1 \mathscr{M}_b \mathscr{M}_w + 4\mathscr{M}_1 \mathcal{L}_k] < 1. \tag{3.3}
$$

Proof. Construct $\mathcal{G} : PC_{1-\eta}(\mathcal{I}, \mathcal{Y}) \to PC_{1-\eta}(\mathcal{I}, \mathcal{Y})$ as

$$
(\mathcal{G}\mathfrak{z})(\mathfrak{t}) = \begin{cases}
S_{p_1,p_2}(\mathfrak{t}) \left[\lambda \int_0^T \mathfrak{z}(\omega)d\omega + c - \mathcal{K}(0,\mathfrak{z}(0)) \right] + \mathcal{K}(\mathfrak{t},\mathfrak{z}(\mathfrak{t})) \\
\quad + \int_0^{\mathfrak{t}} K_{p_1}(\mathfrak{t}-\omega)[B\mathfrak{u}(\omega) + \mathfrak{F}(\omega,\mathfrak{z}(\omega))]d\omega, \qquad \mathfrak{t} \in (0,\mathfrak{t}_1]; \\
\frac{1}{\Gamma(p_1)} \int_{\mathfrak{t}_k}^{\mathfrak{t}} (\mathfrak{t}-\omega)^{p_1-1} \mathscr{I}_k(\omega,\mathfrak{z}(\mathfrak{t}_k^-))d\omega, \qquad \mathfrak{t} \in (\mathfrak{t}_k, \mathfrak{e}_k]; \\
S_{p_1,p_2}(\mathfrak{t}-\mathfrak{e}_k) \left[\frac{1}{\Gamma(p_1)} \int_{\mathfrak{t}_k}^{\mathfrak{e}_k} (\mathfrak{e}_k-\omega)^{p_1-1} \mathscr{I}_k(\omega,\mathfrak{z}(\mathfrak{t}_k^-))d\omega - \mathcal{K}(\mathfrak{e}_k,\mathfrak{z}(\mathfrak{e}_k)) \right] \\
\quad + \mathcal{K}(\mathfrak{t},\mathfrak{z}(\mathfrak{t})) + \int_{\mathfrak{e}_k}^{\mathfrak{t}} K_{p_1}(\mathfrak{t}-\omega)[B\mathfrak{u}(\omega) + \mathfrak{F}(\omega,\mathfrak{z}(\omega))]d\omega, \quad \mathfrak{t} \in (\mathfrak{e}_k, \mathfrak{t}_{k+1}],
\end{cases}
$$

where $\mathfrak{u}(\mathfrak{t})$ is described in (3.1) and (3.2) for $(0, \mathfrak{t}_1]$ and $(\mathfrak{e}_k, \mathfrak{t}_{k+1}]$, respectively. Moreover, by Lemma 3.1, $\mathfrak{z}(\mathfrak{t}_1) = \mathfrak{z}_{\mathfrak{t}_1}$ and $\mathfrak{z}(\mathfrak{t}_{k+1}) = \mathfrak{z}_{\mathfrak{t}_{k+1}}$, $k = 1, 2, \ldots, N$. Let $\aleph_\gamma = \{\mathfrak{z} \in PC_{1-\eta}(\mathcal{I}, \mathcal{Y}) : ||\mathfrak{z}||_{PC_{1-\eta}} \le \gamma\} \subseteq PC_{1-\eta}(\mathcal{I}, \mathcal{Y})$, $\gamma > 0$, and

$$
\gamma > \max \left\{ \mathcal{N} + \mathcal{C}_1(||\mathfrak{z}_{\mathfrak{t}_1}|| + \mathcal{N}), \max_{k=1,2,\ldots,N} \{\mathcal{N}_1 + \mathcal{C}_2(||\mathfrak{z}_{\mathfrak{t}_{k+1}}|| + \mathcal{N}_1)\}, \frac{\mathscr{M}_{\mathscr{I}} T^{p_1}}{\Gamma(p_1 + 1)} \right\}.
$$

Step 1: $\mathcal{G} : \aleph_\gamma \to \aleph_\gamma$.
For $\mathfrak{t} \in (0, \mathfrak{t}_1]$, let $\mathfrak{z} \in \aleph_\gamma$

$$
\begin{aligned}
||(\mathcal{G}\mathfrak{z})(\mathfrak{t})|| \quad &\le \left|\left| S_{p_1,p_2}(\mathfrak{t}) \left[\lambda \int_0^T \mathfrak{z}(\omega)d\omega + c - \mathcal{K}(0,\mathfrak{z}(0)) \right] \right|\right| + \left|\left| \mathcal{K}(\mathfrak{t},\mathfrak{z}(\mathfrak{t})) \right|\right| \\
&\quad + \left|\left| \int_0^{\mathfrak{t}} K_{p_1}(\mathfrak{t}-\omega)\mathfrak{F}(\omega,\mathfrak{z}(\omega))d\omega \right|\right| + \left|\left| \int_0^{\mathfrak{t}} K_{p_1}(\mathfrak{t}-\omega)B\mathfrak{u}(\omega)d\omega \right|\right| \\
&\le \mathscr{M}(\lambda T||\mathfrak{z}|| + |c| + \mu_p) + \mu_p + \frac{\mathscr{M}_1 \mathfrak{t}_1^{p_1}}{p_1} \wp(l)||\Psi||_{L[0,\mathfrak{t}_1]} + \frac{\mathscr{M}_1 \mathscr{M}_b \mathscr{M}_{\mathfrak{u}}^1 \mathfrak{t}^{p_1}}{p_1} \qquad (3.4) \\
&\le \mathcal{N} + \frac{\mathscr{M}_1 \mathscr{M}_b \mathscr{M}_w^1 \mathfrak{t}^{p_1}}{p_1} \left[||\mathfrak{z}_{\mathfrak{t}_1}|| + \mathcal{N} \right] \\
&\le \gamma.
\end{aligned}
$$

Also, for $\mathfrak{t} \in (\mathfrak{e}_k, \mathfrak{t}_{k+1}]$,

$$
\begin{aligned}
||(\mathcal{G}\mathfrak{z})(\mathfrak{t})|| \quad &\leq \left|\left| S_{p_1,p_2}(\mathfrak{t} - \mathfrak{e}_k) \left[\frac{1}{\Gamma(p_1)} \int_{\mathfrak{t}_k}^{\mathfrak{e}_k} (\mathfrak{e}_k - \omega)^{p_1-1} \mathscr{I}_k(\omega, \mathfrak{z}(\mathfrak{t}_k^-)) d\omega - \mathcal{K}(\mathfrak{e}_k, \mathfrak{z}(\mathfrak{e}_k)) \right] \right|\right| \\
&\quad + \left|\left| \mathcal{K}(\mathfrak{t}, \mathfrak{z}(\mathfrak{t})) \right|\right| + \left|\left| \int_{\mathfrak{e}_k}^{\mathfrak{t}} K_{p_1}(\mathfrak{t} - \omega) \mathfrak{F}(\omega, \mathfrak{z}(\omega)) d\omega \right|\right| + \left|\left| \int_{\mathfrak{e}_k}^{\mathfrak{t}} K_{p_1}(\mathfrak{t} - \omega) B\mathfrak{u}(\omega) d\omega \right|\right| \\
&\leq \mathscr{M} \left(\frac{\mathscr{M}_{\mathscr{I}} \mathfrak{t}_{k+1}^{p_1}}{\Gamma(p_1 + 1)} + \mu_p \right) + \mu_p + \frac{\mathscr{M}_1 \mathfrak{t}_1^{p_1}}{p_1} \wp(l) ||\Psi||_{L[0,\mathfrak{t}_{k+1}]} + \frac{\mathscr{M}_1 \mathscr{M}_b \mathscr{M}_\mathfrak{u}^1 T^{p_1}}{p_1} \\
&\leq \mathcal{N}_1 + \frac{\mathscr{M}_1 \mathscr{M}_b \mathscr{M}_w^m T^{p_1}}{p_1} \left(||\mathfrak{z}_{\mathfrak{t}_{k+1}}|| + \mathcal{N}_1 \right) \\
&\leq \gamma.
\end{aligned}
\tag{3.5}
$$

Also, for $\mathfrak{t} \in (\mathfrak{t}_k, \mathfrak{e}_k]$, and $\mathfrak{z} \in \aleph_\gamma$,

$$
||(\mathcal{G}\mathfrak{z})(\mathfrak{t})|| \leq \frac{\mathscr{M}_{\mathscr{I}} \mathfrak{t}_{k+1}^{p_1}}{\Gamma(p_1 + 1)} \leq \gamma.
\tag{3.6}
$$

Hence, from (3.4)–(3.6), for some $\mathfrak{t} \in \mathcal{I}$, gives $||(\mathcal{G}\mathfrak{z})(\mathfrak{t})||_{PC_{1-\eta}} \leq \gamma$. Then $\mathcal{G} : \aleph_\gamma \to \aleph_\gamma$.

Construct $\mathcal{G}_1, \mathcal{G}_2$ as:

$$
(\mathcal{G}_1 y)(\mathfrak{t}) = \begin{cases}
S_{p_1,p_2}(\mathfrak{t}) \left[\lambda \int_0^T \mathfrak{z}(\omega) d\omega + c - \mathcal{K}(0, \mathfrak{z}(0)) \right] + \mathcal{K}(\mathfrak{t}, \mathfrak{z}(\mathfrak{t})), & \mathfrak{t} \in (0, \mathfrak{t}_1], \\
\frac{1}{\Gamma(p_1)} \int_{\mathfrak{t}_k}^{\mathfrak{t}} (\mathfrak{t} - \omega)^{p_1-1} \mathscr{I}_k(\omega, \mathfrak{z}(\mathfrak{t}_k^-)) d\omega, & \mathfrak{t} \in (\mathfrak{t}_k, \mathfrak{e}_k], \ k = 1, 2, \ldots, N, \\
S_{p_1,p_2}(\mathfrak{t} - \mathfrak{e}_k) \left[\frac{1}{\Gamma(p_1)} \int_{\mathfrak{t}_k}^{\mathfrak{e}_k} (\mathfrak{e}_k - \omega)^{p_1-1} \mathscr{I}_k(\omega, \mathfrak{z}(\mathfrak{t}_k^-)) d\omega - \mathcal{K}(\mathfrak{e}_k, \mathfrak{z}(\mathfrak{e}_k)) \right] \\
\quad + \mathcal{K}(\mathfrak{t}, \mathfrak{z}(\mathfrak{t})), & \mathfrak{t} \in (\mathfrak{e}_k, \mathfrak{t}_{k+1}], \ k = 1, 2, \ldots, N,
\end{cases}
$$

and

$$
(\mathcal{G}_2 y)(\mathfrak{t}) = \begin{cases}
\int_0^{\mathfrak{t}} K_{p_1}(\mathfrak{t} - \omega) [B\mathfrak{u}(\omega) + \mathfrak{F}(\omega, \mathfrak{z}(\omega))] d\omega, & \mathfrak{t} \in (0, \mathfrak{t}_1], \\
0, & \mathfrak{t} \in (\mathfrak{t}_k, \mathfrak{e}_k], \\
\int_{\mathfrak{e}_k}^{\mathfrak{t}} K_{p_1}(\mathfrak{t} - \omega) [B\mathfrak{u}(\omega) + \mathfrak{F}(\omega, \mathfrak{z}(\omega))] d\omega, & \mathfrak{t} \in (\mathfrak{e}_k, \mathfrak{t}_{k+1}].
\end{cases}
$$

Clearly, $\mathcal{G} = \mathcal{G}_1 + \mathcal{G}_2$.

Step 2: $\mathcal{G}_1$ is contraction.

Let $\mathfrak{z} \in \aleph_\gamma$, for any $\mathfrak{t} \in (0, \mathfrak{t}_1]$,

$$
\begin{aligned}
||[(\mathcal{G}_1 \mathfrak{z}_1)(\mathfrak{t}) - (\mathcal{G}_1 \mathfrak{z}_2)(\mathfrak{t})]|| \quad &= \left|\left| S_{p_1,p_2}(\mathfrak{t}) \left(\lambda \int_0^T \mathfrak{z}_1(\omega) d\omega - \lambda \int_0^T \mathfrak{z}_2(\omega) d\omega \right) \right|\right| \\
&\quad + ||\mathcal{K}(\mathfrak{t}, \mathfrak{z}_1(\mathfrak{t})) - \mathcal{K}(\mathfrak{t}, \mathfrak{z}_2(\mathfrak{t}))|| \\
&\leq \mathscr{M} \lambda T ||\mathfrak{z}_1(\omega) - \mathfrak{z}_2(\omega)||_{PC_{1-\eta}} + \mathcal{L}_p ||\mathfrak{z}_1(\omega) - \mathfrak{z}_2(\omega)||_{PC_{1-\eta}} \\
&\leq (\mathscr{M} \lambda T + \mathcal{L}_p) ||\mathfrak{z}_1 - \mathfrak{z}_2||_{PC_{1-\eta}} \\
&\leq k_1 ||\mathfrak{z}_1 - \mathfrak{z}_2||_{PC_{1-\eta}}.
\end{aligned}
\tag{3.7}
$$

Also, for $\mathfrak{z} \in \aleph_\gamma$, $\mathfrak{t} \in (\mathfrak{e}_k, \mathfrak{t}_{k+1}]$,

$$
\begin{aligned}
||(\mathcal{G}_1 \mathfrak{z}_1)(\mathfrak{t}) - (\mathcal{G}_1 \mathfrak{z}_2)(\mathfrak{t})|| &\leq \left(\mathscr{M} \left(\frac{K_{\mathscr{I}_k} \mathfrak{t}_{k+1}^{p_1}}{\Gamma(p_1 + 1)} + \mathcal{L}_p \right) + \mathcal{L}_p \right) ||\mathfrak{z}_1 - \mathfrak{z}_2||_{PC_{1-\eta}} \\
&\leq k_1 ||\mathfrak{z}_1 - \mathfrak{z}_2||_{PC_{1-\eta}}.
\end{aligned}
\tag{3.8}
$$

Also, for $\mathfrak{t} \in (\mathfrak{t}_k, \mathfrak{e}_k]$, and $\mathfrak{z} \in \aleph_\gamma$,

$$
||(\mathcal{G}_1 \mathfrak{z}_1)(\mathfrak{t}) - (\mathcal{G}_1 \mathfrak{z}_2)(\mathfrak{t})|| \leq \frac{K_{\mathscr{I}_k} \mathfrak{t}_{k+1}^{p_1}}{\Gamma(p_1 + 1)} ||\mathfrak{z}_1 - \mathfrak{z}_2||_{PC_{1-\eta}} \leq k_1 ||\mathfrak{z}_1 - \mathfrak{z}_2||_{PC_{1-\eta}}.
\tag{3.9}
$$

For any $\mathfrak{t} \in \mathcal{I}$, $||(\mathcal{G}_1 \mathfrak{z}_1)(\mathfrak{t}) - (\mathcal{G}_1 \mathfrak{z}_2)(\mathfrak{t})|| \leq k_1 ||\mathfrak{z}_1 - \mathfrak{z}_2||_{PC_{1-\eta}}$. Since $k_1 < 1$, $\mathcal{G}_1$ is contracting operator.

Step 3: By step 1, it is clear that $\mathcal{G}_2$ is bounded. To prove continuity, consider a sequence $\{\mathfrak{z}^n\}_{n=1}^\infty$ in $\aleph_\gamma$ such that $\mathfrak{z}^n \to \mathfrak{z}$ in $\aleph_\gamma$. For $\mathfrak{t} \in (0, \mathfrak{t}_1]$,

$$
\begin{aligned}
&||(\mathcal{G}_2 \mathfrak{z}^n)(\mathfrak{t}) - (\mathcal{G}_2 \mathfrak{z})(\mathfrak{t})|| \\
&\leq \left|\left| \mathfrak{t}^{\eta-1} \int_0^{\mathfrak{t}} (\mathfrak{t} - \omega)^{p_1 - 1} [\mathfrak{F}(\omega, \mathfrak{z}^n(\omega)) - \mathfrak{F}(\omega, \mathfrak{z}(\omega))] d\omega \right|\right| \\
&+ \left|\left| \mathfrak{t}^{\eta-1} \int_0^{\mathfrak{t}} (\mathfrak{t} - \omega)^{p_1 - 1} [B\mathfrak{u}_{\mathfrak{z}^n}(\omega) - B\mathfrak{u}_{\mathfrak{z}}(\omega)] d\omega \right|\right| \\
&\leq \mathscr{M}_1 \mathcal{L}_f \int_0^{\mathfrak{t}} ||\mathfrak{z}^n(\omega) - \mathfrak{z}(\omega)|| d\omega \\
&+ \mathscr{M}_1 \mathscr{M}_b \mathscr{M}_w^1 \int_0^{\mathfrak{t}} \left[\left|\left| S_{p_1, p_2}(\mathfrak{t})(\lambda \int_0^T \mathfrak{z}^n(\omega) d\omega - \lambda \int_0^T \mathfrak{z}(\omega) d\omega) \right|\right| \right. \\
&+ ||\mathcal{K}(\mathfrak{t}_1, \mathfrak{z}^n(\mathfrak{t}_1)) - \mathcal{K}(\mathfrak{t}, \mathfrak{z}(\mathfrak{t}))|| + \int_0^{\mathfrak{t}_1} (\mathfrak{t} - \omega)^{p_1 - 1} ||\mathfrak{F}(\omega, \mathfrak{z}^n(\omega)) d\omega - \mathfrak{F}(\omega, \mathfrak{z}(\omega))|| d\omega \Big] d\tau \\
&\leq \mathscr{M}_1 \mathcal{L}_f \frac{\mathfrak{t}_1^{p_1}}{p_1} ||\mathfrak{z}^n - \mathfrak{z}||_{PC_{1-\eta}} + \mathcal{C}_1 \left[\mathcal{L}_p + \mathscr{M}_1 \mathcal{L}_f \frac{\mathfrak{t}_1^{p_1}}{p_1} \right] ||\mathfrak{z}^n - \mathfrak{z}||_{PC_{1-\eta}}.
\end{aligned}
\tag{3.10}
$$

Therefore, $||(\mathcal{G}_2 \mathfrak{z}^n)(\mathfrak{t}) - (\mathcal{G}_2 \mathfrak{z})(\mathfrak{t})|| \to 0$ as $n \to \infty$. Also, for $\mathfrak{t} \in (\mathfrak{e}_k, \mathfrak{t}_{k+1}]$, $k = 1, 2, \ldots, N$,

$$
\begin{aligned}
&||(\mathcal{G}_2 \mathfrak{z}^n)(\mathfrak{t}) - (\mathcal{G}_2 \mathfrak{z})(\mathfrak{t})|| \\
&\leq \left|\left| \mathfrak{t}^{1-\eta} \int_{\mathfrak{e}_k}^{\mathfrak{t}} (\mathfrak{t} - \omega)^{p_1 - 1} [\mathfrak{F}(\omega, \mathfrak{z}^n(\omega)) - \mathfrak{F}(\omega, \mathfrak{z}(\omega))] d\omega \right|\right| \\
&+ \left|\left| \mathfrak{t}^{1-\eta} \int_{\mathfrak{e}_k}^{\mathfrak{t}} (\mathfrak{t} - \omega)^{p_1 - 1} [B\mathfrak{u}_{\mathfrak{z}^n}(\omega) - B\mathfrak{u}_{\mathfrak{z}}(\omega)] d\omega \right|\right| \\
&\leq \mathscr{M}_1 \mathcal{L}_f \int_{\mathfrak{e}_k}^{\mathfrak{t}} ||\mathfrak{z}^n(\omega) - \mathfrak{z}(\omega)|| d\omega \\
&+ \mathscr{M}_1 \mathscr{M}_b \mathscr{M}_w^m \int_{\mathfrak{e}_k}^{\mathfrak{t}} \left(\left|\left| S_{p_1, p_2}(\mathfrak{t}_{k+1} - \mathfrak{e}_k) \left[\frac{1}{\Gamma(p_1)} \int_{\mathfrak{t}_k}^{\mathfrak{e}_k} (\mathfrak{e}_k - \omega)^{p_1 - 1} \right. \right.\right. \right. \\
&\times [\mathscr{I}_k(\omega, \mathfrak{z}^n(\mathfrak{t}_k^-)) - \mathscr{I}_k(\omega, \mathfrak{z}(\mathfrak{t}_k^-))] d\omega + [\mathcal{K}(\mathfrak{e}_k, \mathfrak{z}^n(\mathfrak{e}_k)) - \mathcal{K}(\mathfrak{e}_k, \mathfrak{z}(\mathfrak{e}_k))] \Big] \Big|\Big| \\
&+ ||\mathcal{K}(\mathfrak{t}, \mathfrak{z}^n(\mathfrak{t})) - \mathcal{K}(\mathfrak{t}, \mathfrak{z}(\mathfrak{t}))|| + \int_{\mathfrak{e}_k}^{\mathfrak{t}_{k+1}} (\mathfrak{t}_{k+1} - \omega)^{p_1 - 1} ||\mathfrak{F}(\omega, \mathfrak{z}^n(\omega)) d\omega - \mathfrak{F}(\omega, \mathfrak{z}(\omega))|| d\omega \Bigg) d\tau \\
&\leq \mathscr{M}_1 \mathcal{L}_f \frac{\mathfrak{t}_{k+1}^{p_1}}{p_1} ||\mathfrak{z}^n - \mathfrak{z}||_{PC_{1-\eta}} \\
&+ \mathcal{C}_2 \left(\mathscr{M} \left(\frac{K_{\mathscr{I}_k} \mathfrak{t}_{k+1}^{p_1}}{\Gamma(p_1 + 1)} + \mathcal{L}_p \right) + \mathcal{L}_p + \mathscr{M}_1 \mathcal{L}_f \frac{\mathfrak{t}_{k+1}^{p_1}}{p_1} \right) ||\mathfrak{z}^n - \mathfrak{z}||_{PC_{1-\eta}}.
\end{aligned}
\tag{3.11}
$$

Hence, $||(\mathcal{G}_2\mathfrak{z}^n)(\mathfrak{t}) - (\mathcal{G}_2\mathfrak{z})(\mathfrak{t})||$ approaches to 0 as n approaches to ∞. Hence from (3.10) and (3.11) and for each $\mathfrak{t} \in \mathcal{I}$, $||(\mathcal{G}_2\mathfrak{z}^n)(\mathfrak{t}) - (\mathcal{G}_2\mathfrak{z})(\mathfrak{t})|| \to 0$ as $n \to \infty$.

Step 4: $\mathcal{G}_2$ is equicontinuous.

Take $\tau_1 < \tau_2$ on $\aleph_\gamma$, and for $\tau_1, \tau_2 \in (0, \mathfrak{t}_1]$,

$$
\begin{aligned}
&||(\mathcal{G}_2\mathfrak{z})(\tau_2) - (\mathcal{G}_2\mathfrak{z})(\tau_1)|| \\
&\leq \left|\left| \tau_2^{\eta-1} \int_0^{\tau_2} (\tau_2 - \omega)^{p_1-1}[\mathfrak{F}(\omega, \mathfrak{z}(\omega)) + B\mathfrak{u}(\omega)]d\omega \right.\right. \\
&\quad \left.\left. -\tau_1^{\eta-1} \int_0^{\tau_1} (\tau_1 - \omega)^{p_1-1}[\mathfrak{F}(\omega, \mathfrak{z}(\omega)) + B\mathfrak{u}(\omega)]d\omega \right|\right| \\
&\leq \mathcal{M}_1 \mathcal{L}_f \left(\int_0^{\tau_1} [\tau_2^{\eta-1}(\tau_2 - \omega)^{p_1-1} - \tau_1^{\eta-1}(\tau_1 - \omega)^{p_1-1}]d\omega + \int_{\tau_1}^{\tau_2} \tau_2^{\eta-1}(\tau_2 - \omega)^{p_1-1}d\omega \right) \\
&\quad + \mathcal{M}_1 \mathcal{M}_b \mathcal{M}_w^1 (||\mathfrak{z}_{\mathfrak{t}_1}|| + \mathcal{N}) \left(\int_0^{\tau_1} [\tau_2^{\eta-1}(\tau_2 - \omega)^{p_1-1} - \tau_1^{\eta-1}(\tau_1 - \omega)^{p_1-1}]d\omega \right. \\
&\quad \left. + \int_{\tau_1}^{\tau_2} \tau_2^{\eta-1}(\tau_2 - \omega)^{p_1-1}d\omega \right).
\end{aligned}
\tag{3.12}
$$

Similarly, For $\tau_1, \tau_2 \in (\mathfrak{e}_k, \mathfrak{t}_{k+1}]$,

$$
\begin{aligned}
&||(\mathcal{G}_2\mathfrak{z})(\tau_2) - (\mathcal{G}_2\mathfrak{z})(\tau_1)|| \\
&\leq \left|\left| \tau_2^{\eta-1} \int_{\mathfrak{e}_k}^{\tau_2} (\tau_2 - \omega)^{p_1-1}[\mathfrak{F}(\omega, \mathfrak{z}(\omega)) + B\mathfrak{u}(\omega)]d\omega \right.\right. \\
&\quad \left.\left. - \int_{\mathfrak{e}_k}^{\tau_1} (\tau_1 - \omega)^{p_1-1}[\mathfrak{F}(\omega, \mathfrak{z}(\omega)) + B\mathfrak{u}(\omega)]d\omega \right|\right| \\
&\leq \mathcal{M}_1 \mathcal{L}_f \left(\int_{\mathfrak{e}_k}^{\tau_1} [\tau_2^{\eta-1}(\tau_2 - \omega)^{p_1-1} - \tau_1^{\eta-1}(\tau_1 - \omega)^{p_1-1}]d\omega + \int_{\tau_1}^{\tau_2} \tau_2^{\eta-1}(\tau_2 - \omega)^{p_1-1}d\omega \right) \\
&\quad + \mathcal{M}_1 \mathcal{M}_b \mathcal{M}_w^m (||\mathfrak{z}_{\mathfrak{t}_{k+1}}|| + \mathcal{N}_1) \left(\int_{\mathfrak{e}_k}^{\tau_1} [\tau_2^{\eta-1}(\tau_2 - \omega)^{p_1-1} - \tau_1^{\eta-1}(\tau_1 - \omega)^{p_1-1}]d\omega \right. \\
&\quad \left. + \int_{\tau_1}^{\tau_2} \tau_2^{\eta-1}(\tau_2 - \omega)^{p_1-1}d\omega \right).
\end{aligned}
\tag{3.13}
$$

By $(H3)$, $||(\mathcal{G}_2\mathfrak{z})(\tau_2) - (\mathcal{G}_2\mathfrak{z})(\tau_1)|| \to 0$ as $\tau_2 \to \tau_1$. Then $\mathcal{G}_2$ is equicontinuous.

The countable subset $D_0 = \{\mathfrak{z}_n\}_{n=1}^\infty \subset D$, and by Lemma 2.4, we have

$$
\ell(\mathcal{G}_2(D))_{PC_{1-\eta}} \leq 2\, \ell(\mathcal{G}_2(D_0))_{PC_{1-\eta}},
\tag{3.14}
$$

where D is a bounded subset of $\aleph_\gamma$. Since $\mathcal{G}_2(D_0) \subset \mathcal{G}_2(\aleph_\gamma)$ is bounded and equicontinuous, by Lemma 2.6,

$$
\ell(\mathcal{G}_2(D_0))_{PC_{1-\eta}} \leq \max_{\mathfrak{t} \in (\mathfrak{e}_k, \mathfrak{t}_{k+1}],\ k=0,1,2,\ldots,N} \ell(\mathcal{G}_2(D_0))_{PC_{1-\eta}}(\mathfrak{t}).
\tag{3.15}
$$

Moreover, for $\mathfrak{t} \in (\mathfrak{e}_k, \mathfrak{t}_{k+1}]$, $(H4)$, $(H7)$ and $\mathcal{G}_2$, with Lemma 2.5, we have

$$
\begin{aligned}
\ell(\mathcal{G}_2(D_0))(\mathfrak{t}) \quad &\leq \ell\left(\mathscr{M}_1 \int_{\mathfrak{e}_k}^{\mathfrak{t}} \left[B\mathfrak{u}(\omega, \{\mathfrak{z}_n(\omega)\}_{n=1}^{\infty}) + \mathfrak{F}(\omega, \{\mathfrak{z}_n(\omega)\}_{n=1}^{\infty})\right] d\omega\right) \\
&\leq \left[2\mathcal{L}_{\mathfrak{u}}^* \sigma^* \mathscr{M}_1 \mathscr{M}_b \mathscr{M}_w + 2\mathscr{M}_1 \mathcal{L}_k\right] \int_{\mathfrak{e}_k}^{\mathfrak{t}} \ell(\{\mathfrak{z}_n(\omega)\}_{n=1}^{\infty}) d\omega \\
&\leq \left[2\mathcal{L}_{\mathfrak{u}}^* \sigma^* \mathscr{M}_1 \mathscr{M}_b \mathscr{M}_w + 2\mathscr{M}_1 \mathcal{L}_k\right] \ell(D)_{PC_{1-\eta}} (\mathfrak{t}_{k+1} - \mathfrak{e}_k).
\end{aligned}
\tag{3.16}
$$

Then, by (3.14)–(3.16) and $(H2)$,

$$
\begin{aligned}
\ell(\mathcal{G}_2(D)(\mathfrak{t}))_{PC_{1-\eta}} \quad &\leq \left[2\mathcal{L}_{\mathfrak{u}}^* \sigma^* \mathscr{M}_1 \mathscr{M}_b \mathscr{M}_w + 2\mathscr{M}_1 \mathcal{L}_k\right] \ell(D)_{PC_{1-\eta}} (\mathfrak{t}_{k+1} - \mathfrak{e}_k) \\
&\leq \left[4\mathcal{L}_{\mathfrak{u}}^* \sigma^* \mathscr{M}_1 \mathscr{M}_b \mathscr{M}_w + 4\mathscr{M}_1 \mathcal{L}_k\right] \ell(D)_{PC_{1-\eta}}.
\end{aligned}
\tag{3.17}
$$

Now, for any $\mathfrak{t} \in (\mathfrak{e}_k, \mathfrak{t}_{k+1}]$, on $D \in \aleph_\gamma$,

$$
\ell(\mathcal{G}_1(D)) \leq \left[\mathscr{M}\lambda + 2\mathcal{L}_p^* + \mathscr{M}_{\mathscr{I}}(\mathscr{M}+1)\right]\ell(D).
\tag{3.18}
$$

Also,

$$
\begin{aligned}
\ell(\mathcal{G}(D)) &\leq \ell(\mathcal{G}_1(D)) + \ell(\mathcal{G}_2(D)) \\
&\leq \left[\mathscr{M}\lambda + 2\mathcal{L}_p^* + \mathscr{M}_{\mathscr{I}}(\mathscr{M}+1) + 4\mathcal{L}_{\mathfrak{u}}^* \sigma^* \mathscr{M}_1 \mathscr{M}_b \mathscr{M}_w + 4\mathscr{M}_1 \mathcal{L}_k\right]\ell(D)_{PC_{1-\eta}}.
\end{aligned}
\tag{3.19}
$$

Combining Lemma 3.1, and (3.3) and (3.19) it is clear that the mapping $\mathcal{G}$ from $\aleph_\gamma$ to $\aleph_\gamma$ is κ-set-contractive. Hence, the system $\mathcal{G}$ has a fixed point by Lemma 3.2. This completes the proof.

4 Optimal control

$(H8)$ (i) The Lagrange function $\mathfrak{L} : \mathcal{I} \times \mathcal{Y} \times U \to \mathbb{R} \cup \{\infty\}$ is Borel measurable;
 (ii) For $\mathfrak{t} \in \mathcal{I}$, and for every $\mathfrak{z}_1, \mathfrak{z}_2 \in \mathcal{Y}$, $\mathfrak{L}(\mathfrak{t}, \mathfrak{z}, \cdot)$ is convex on U;
 (iii) For almost all $\mathfrak{t} \in \mathcal{I}$, $\mathfrak{L}(\mathfrak{t}, \cdot, \cdot)$ is sequentially lower semi continuous on $\mathcal{Y} \times U$;
 (iv) For $\mathfrak{c}_1 \geq 0$, $\mathfrak{c}_2 > 0$, $\mathfrak{h} \in L^P(\mathcal{I}, \mathbb{R})$,

$$
\mathfrak{L}(\mathfrak{t}, \mathfrak{z}, \mathfrak{u}) \geq \mathfrak{h}(\mathfrak{t}) + \mathfrak{c}_1 \|\mathfrak{z}\|_{PC_{1-\eta}} + \mathfrak{c}_2 \|\mathfrak{u}\|^P.
$$

This part deals with the verification of existence of optimal pair for the system (1.1) by sequencing technique as discussed in [46, 48]. Let the cost function($\mathfrak{L}$) as:

$$
\mathfrak{I}(\mathfrak{z}^{\mathfrak{u}}, \mathfrak{u}) = \int_0^T \mathfrak{L}(\mathfrak{t}, \mathfrak{z}(\mathfrak{t}), \mathfrak{u}(\mathfrak{t})) d\mathfrak{t}, \quad \mathfrak{u} \in \mathcal{U}_{ad}.
$$

Define the admissible control function $\mathcal{U}_{ad}$ as:

$$
\mathcal{U}_{ad} = \{\mathfrak{u} \in L^P(\mathcal{I}, H); \mathfrak{u}(\mathfrak{t}) \in \wp(\mathfrak{t}), a.e. \, \mathfrak{t} \in \mathcal{I}\}, \quad P > 1,
$$

where $\mathfrak{u}(\mathfrak{t})$ takes its values in $\mathcal{S} \subset U$. A multivalued map $\wp : \mathcal{I} \to PC_{1-\eta}$, is measurable as $\wp(\cdot) \subset \mathcal{S}$. It is clear that $\mathcal{U}_{ad}$ is bounded, convex & closed with $\mathcal{U}_{ad} = 0$. Define the solution set

$$
\mathcal{T}(\mathfrak{u}) = \{\mathfrak{z}^{\mathfrak{u}} \in \aleph_\gamma : \mathfrak{z}^{\mathfrak{u}} \, \mathfrak{u} \in \mathcal{U}_{ad}\}.
$$

Also, the set of all $A_{ad} = \{(\mathfrak{z}^{\mathfrak{u}}, \mathfrak{u}); \mathfrak{u} \in \mathcal{U}_{ad}; \mathfrak{z}^{\mathfrak{u}} \in \mathcal{T}(\mathfrak{u})\}$.

Theorem 4.1. *The system* (1.1) *is optimal controllable together with the assumptions (H1)-(H8) provided*

$$\mathfrak{I}(\tilde{\mathfrak{z}}^{\mathfrak{u}^0}, \mathfrak{u}^0) = \int_0^T \mathfrak{L}(\mathfrak{t}, \tilde{\mathfrak{z}}^{\mathfrak{u}^0}(\mathfrak{t}), \mathfrak{u}^0(\mathfrak{t}))d\mathfrak{t} \leq \mathfrak{I}(\mathfrak{z}^{\mathfrak{u}}, \mathfrak{u}), \ \forall \ (\mathfrak{z}^{\mathfrak{u}}, \mathfrak{u}) \in A_{ad}.$$

Proof. Define $\mathfrak{I}(\mathfrak{u}) = \inf_{\mathfrak{z}^{\mathfrak{u}} \in \mathcal{T}(\mathfrak{u})} \mathfrak{I}(\mathfrak{z}^{\mathfrak{u}}, \mathfrak{u})$. Initially we prove $\mathfrak{I}(\overline{\mathfrak{z}}^{\mathfrak{u}}, \mathfrak{u}) = \mathfrak{I}(\mathfrak{u})$, $\mathfrak{z}^{\mathfrak{u}} \in \mathcal{T}(\mathfrak{u})$. If $\mathfrak{I}(\mathfrak{u}) = +\infty$ or $\mathcal{T}(\mathfrak{u})$ has finite elements, the proof is trivial. Using $(H8)(iv)$, $\mathfrak{I}(\mathfrak{u}) > -\infty$. Let $\mathfrak{I}(\mathfrak{u}) < \infty$. By infimum properties, a sequence $\{\mathfrak{z}_n^{\mathfrak{u}}\}_{n=1}^\infty \in \mathcal{T}(\mathfrak{u})$ satisfies $\mathfrak{I}(\mathfrak{z}_n^{\mathfrak{u}}, \mathfrak{u}) \to \mathfrak{I}(\mathfrak{u})$ as $n \to \infty$. Using reflexive property, $\{\mathfrak{u}^0\} \in \mathcal{T}(\mathfrak{u})$ provided $\mathfrak{u}^0 \in \mathcal{U}_{ad}$.

For $n \geq 1$,

$$(\mathfrak{z}_n^{\mathfrak{u}})(\mathfrak{t}) = \begin{cases} S_{p_1,p_2}(\mathfrak{t})\left[\lambda \int_0^T \mathfrak{z}_n^{\mathfrak{u}}(\omega)d\omega + c - \mathcal{K}(0, \mathfrak{z}(0))\right] + \mathcal{K}(\mathfrak{t}, \mathfrak{z}_n^{\mathfrak{u}}(\mathfrak{t})) \\ + \int_0^{\mathfrak{t}} K_{p_1}(\mathfrak{t} - \omega)[B\mathfrak{u}(\omega) + \mathfrak{F}(\omega, \mathfrak{z}_n^{\mathfrak{u}}(\omega))]d\omega, \qquad \mathfrak{t} \in (0, \mathfrak{t}_1]; \\ S_{p_1,p_2}(\mathfrak{t} - \mathfrak{e}_k)\left[\frac{1}{\Gamma(p_1)}\int_{\mathfrak{t}_k}^{\mathfrak{e}_k}(\mathfrak{e}_k - \omega)^{p_1-1}\mathscr{I}_k(\omega, \mathfrak{z}_n^{\mathfrak{u}}(\mathfrak{t}_k^-))d\omega - \mathcal{K}(\mathfrak{e}_k, \mathfrak{z}_n^{\mathfrak{u}}(\mathfrak{e}_k))\right] \\ + \mathcal{K}(\mathfrak{t}, \mathfrak{z}_n^{\mathfrak{u}}(\mathfrak{t})) + \int_{\mathfrak{e}_k}^{\mathfrak{t}} K_{p_1}(\mathfrak{t} - \omega)[B\mathfrak{u}(\omega) + \mathfrak{F}(\omega, \mathfrak{z}_n^{\mathfrak{u}}(\omega))]d\omega, \ \mathfrak{t} \in (\mathfrak{e}_k, \mathfrak{t}_{k+1}], \end{cases}$$

where

$$(\mathfrak{z}_n^{\mathfrak{u}})(\mathfrak{t}) = (\mathcal{G}_1\mathfrak{z}_n^{\mathfrak{u}})(\mathfrak{t}) + (\mathcal{G}_2\mathfrak{z}_n^{\mathfrak{u}})(\mathfrak{t}) = (\mathcal{G}\mathfrak{z}_n^{\mathfrak{u}})(\mathfrak{t}).$$

To prove $(\mathcal{B}\mathfrak{z}_n^{\mathfrak{u}})(\mathfrak{t}) : \{\mathcal{G}_2(\mathfrak{t}); \ \mathfrak{z}_n^{\mathfrak{u}} \in \aleph\gamma\}$ is relatively compact in $PC_{1-\eta}$ for each $\mathfrak{t} \in \mathcal{I}$.

It is clear that $\mathcal{B}(0) : \{\mathcal{G}_2(0); \ \mathfrak{z}_n^{\mathfrak{u}} \in \aleph\gamma\}$ is relatively compact. For any $\mathfrak{u} \in U, \ \mathfrak{t} \in \mathcal{I}, \ \mathfrak{z}_n^{\mathfrak{u}} \in \aleph_\gamma$,

$$(\mathcal{G}_2\mathfrak{z}_n^{\mathfrak{u}})(\mathfrak{t}) = \int_0^{\mathfrak{t}} K_{p_1}(\mathfrak{t} - \omega)[B\mathfrak{u}(\omega) + \mathfrak{F}(\omega, \mathfrak{z}_n^{\mathfrak{u}}(\omega))]d\omega.$$

By $(H3)$, and the property of admissible of control functions the set $\mathcal{W}_\epsilon = \{K_{p_1}(\mathfrak{t} - \omega)$ $[B\mathfrak{u}(\omega) + \mathfrak{F}(\omega, \mathfrak{z}_n^{\mathfrak{u}}(\omega))]; 0 \leq \mathfrak{e}_k \leq \mathfrak{t}_k - \epsilon\}$ is relatively compact. Therefore, $\overline{\mathcal{W}_\epsilon}$, the convex hull of $\mathcal{W}_\epsilon$ is compact due to Lemma 2.3(ii). Using Lemma 2.5, we can conclude $(\mathcal{G}_2^\epsilon\mathfrak{z}_n^{\mathfrak{u}})(\mathfrak{t}) \in \overline{\mathcal{W}_\epsilon}$ for all $\mathfrak{t} \in \mathcal{I}$. Therefore $\mathcal{B}_\epsilon(\mathfrak{t}) : \{(\mathcal{G}_2^\epsilon\mathfrak{z}_n^{\mathfrak{u}})(\mathfrak{t}); \ \mathfrak{z}_n^{\mathfrak{u}} \in \aleph\gamma\}$ is relatively compact in $PC_{1-\eta}$. For $\mathfrak{t} \in (0, \mathfrak{t}_1]$,

$$||(\mathcal{G}_2\mathfrak{z}_n^{\mathfrak{u}})(\mathfrak{t}) - (\mathcal{G}_2^\epsilon\mathfrak{z}_n^{\mathfrak{u}})(\mathfrak{t})||$$

$$\leq \left|\left|\int_0^{\mathfrak{t}} K_{p_1}(\mathfrak{t} - \omega)[\mathfrak{F}(\omega, \mathfrak{z}_n^{\mathfrak{u}}(\omega)) + B\mathfrak{u}(\omega)]d\omega - \int_0^{\mathfrak{t}-\epsilon}[\mathfrak{F}(\omega, \mathfrak{z}_n^{\mathfrak{u}}(\omega)) + B\mathfrak{u}(\omega)]d\omega\right|\right|$$

$$\leq \int_{\mathfrak{t}-\epsilon}^{\mathfrak{t}} K_{p_1}(\mathfrak{t} - \omega)[\mathfrak{F}(\omega, \mathfrak{z}_n^{\mathfrak{u}}(\omega)) + B\mathfrak{u}(\omega)]d\omega$$

$$\leq \mathscr{M}_1\mathcal{L}_f \int_{\mathfrak{t}-\epsilon}^{\mathfrak{t}}||\mathfrak{z}_n^{\mathfrak{u}}(\omega)||d\omega + \mathscr{M}_1\mathscr{M}_b||\mathfrak{u}||_{\mathcal{L}_p}.$$

Similarly, for $t \in (\mathfrak{e}_k, t_{k+1}]$,

$$
\begin{aligned}
&||(\mathcal{G}_2 \mathfrak{z}_n^{\mathfrak{u}})(t) - (\mathcal{G}_2^{\epsilon} \mathfrak{z}_n^{\mathfrak{u}})(t)|| \\
&\leq \left\| \int_{\mathfrak{e}_k}^{t} K_{p_1}(t - \omega)[\mathfrak{F}(\omega, \mathfrak{z}_n^{\mathfrak{u}}(\omega)) + B\mathfrak{u}(\omega)]d\omega - \int_{\mathfrak{e}_k}^{t-\epsilon}[\mathfrak{F}(\omega, \mathfrak{z}_n^{\mathfrak{u}}(\omega)) + B\mathfrak{u}(\omega)]d\omega \right\| \\
&\leq \int_{t-\epsilon}^{t} K_{p_1}(t - \omega)[\mathfrak{F}(\omega, \mathfrak{z}_n^{\mathfrak{u}}(\omega)) + B\mathfrak{u}(\omega)]d\omega \\
&\leq \mathscr{M}_1 \mathcal{L}_f \int_{t-\epsilon}^{t} ||\mathfrak{z}_n^{\mathfrak{u}}(\omega)||d\omega + \mathscr{M}_1 \mathscr{M}_b ||\mathfrak{u}||_{\mathcal{L}_p},
\end{aligned}
$$

implies that $\lim \epsilon \to 0 ||(\mathcal{G}_2 \mathfrak{z}_n^{\mathfrak{u}})(t) - (\mathcal{G}_2^{\epsilon} \mathfrak{z}_n^{\mathfrak{u}})(t)|| = 0$. Hence $\mathcal{B}_\epsilon(t)$, is a family of relatively compact sets. Moreover, $\mathcal{G}_1 \mathfrak{z}_n^{\mathfrak{u}}$ is bounded and equicontinuous in $\aleph_\gamma$. By (3.16) and (3.18) we have

$$
\ell(\mathfrak{z}_n^{\mathfrak{u}}) \leq [\mathscr{M}\lambda + 2\mathcal{L}_p^* + \mathscr{M}_{\mathscr{I}}(\mathscr{M} + 1) + 4\mathcal{L}_{\mathfrak{u}}^*\sigma^*\mathscr{M}_1\mathscr{M}_b\mathscr{M}_w + 4\mathscr{M}_1\mathcal{L}_k]\ell(\mathfrak{z}_n^{\mathfrak{u}}),
$$

leads to $\ell(\{\mathfrak{z}_n^{\mathfrak{u}}\}_{n=0}^{\infty}) = 0$ by using (3.3). Hence, $\{\mathfrak{z}_n^{\mathfrak{u}}\}_{n=1}^{\infty}$ is relatively compact in $PC_{1-\eta}$. Assume $\tilde{\mathfrak{z}}^{\mathfrak{u}}$, a subsequence in $PC_{1-\eta}$ of $\{\mathfrak{z}_n^{\mathfrak{u}}\}_{n=0}^{\infty}$ such that $\mathfrak{z}_n^{\mathfrak{u}} \to \tilde{\mathfrak{z}}^{\mathfrak{u}}$ as $\lim n \to \infty$. Moreover, by Lebesgue theorem and (H1), (H3), (H5)

$$
(\tilde{\mathfrak{z}}^{\mathfrak{u}})(t) = \begin{cases} S_{p_1, p_2}(t)\left[\lambda \int_0^T \tilde{\mathfrak{z}}^{\mathfrak{u}}(\omega)d\omega + c - \mathcal{K}(0, \mathfrak{z}(0))\right] + \mathcal{K}(t, \tilde{\mathfrak{z}}^{\mathfrak{u}}(t)) \\ + \int_0^t K_{p_1}(t - \omega)[B\mathfrak{u}(\omega) + \mathfrak{F}(\omega, \tilde{\mathfrak{z}}^{\mathfrak{u}}(\omega))]d\omega, \qquad t \in (0, t_1]; \\ S_{p_1, p_2}(t - \mathfrak{e}_k)\left[\frac{1}{\Gamma(p_1)} \int_{t_k}^{\mathfrak{e}_k}(\mathfrak{e}_k - \omega)^{p_1-1}\mathscr{I}_k(\omega, \tilde{\mathfrak{z}}^{\mathfrak{u}}(t_k^-))d\omega - \mathcal{K}(\mathfrak{e}_k, \tilde{\mathfrak{z}}^{\mathfrak{u}}(\mathfrak{e}_k))\right] \\ + \mathcal{K}(t, \tilde{\mathfrak{z}}^{\mathfrak{u}}(t)) + \int_{\mathfrak{e}_k}^t K_{p_1}(t - \omega)[B\mathfrak{u}(\omega) + \mathfrak{F}(\omega, \tilde{\mathfrak{z}}^{\mathfrak{u}}(\omega))]d\omega, \quad t \in (\mathfrak{e}_k, t_{k+1}]. \end{cases}
$$

Then, $\tilde{\mathfrak{z}}^{\mathfrak{u}} \in \mathcal{T}(\mathfrak{u})$ is continuously embedded in $L^1(\mathcal{I}, U)$, by Balder's theorem [49] and (H8),

$$
\mathfrak{I}(\mathfrak{u}) = \lim_{n \to \infty} \int_0^T \mathfrak{L}(t, \mathfrak{z}_n^{\mathfrak{u}}(t), \mathfrak{u}(t))dt \geq \int_0^T \mathfrak{L}(t, \tilde{\mathfrak{z}}^{\mathfrak{u}}(t), \mathfrak{u}(t))dt = \mathfrak{I}(\tilde{\mathfrak{z}}^{\mathfrak{u}}, \mathfrak{u}) \geq \mathfrak{I}(\mathfrak{u}),
$$

which shows $\mathfrak{I}(\tilde{\mathfrak{z}}^{\mathfrak{u}}, \mathfrak{u}) \to \mathfrak{I}(\mathfrak{u})$. Therefore, $\mathfrak{I}(\mathfrak{u})$ reaches its least value at $\tilde{\mathfrak{z}}^{\mathfrak{u}} \in \mathcal{T}(\mathfrak{u})$ for every $\mathfrak{u} \in \mathcal{U}_{ad}$.

Also, consider $\mathfrak{u}^0 \in \mathcal{U}_{ad}$ such that $\mathfrak{I}(\mathfrak{u}) = \inf_{\mathfrak{u} \in \mathcal{U}_{ad}} \mathfrak{I}(\mathfrak{u})$. By the infimum property, $\{\mathfrak{u}_n\}_{n=0}^{\infty} \subseteq \mathcal{U}_{ad}$ provided $\lim_{n \to \infty} \mathfrak{I}(\mathfrak{u}_n) = \inf_{\mathfrak{u} \in \mathcal{U}_{ad}} \mathfrak{I}(\mathfrak{u})$. Since $\{\mathfrak{u}_n\}_{n=0}^{\infty}$ in $\mathcal{L}_P(\mathcal{I}, U)$ is bounded for $P > 1$, $\mathfrak{u}^0 \in \mathcal{L}_P(\mathcal{I}, U)$ and by relative compactness of $\mathfrak{z}_n^{\mathfrak{u}}$ there is a subsequence $\tilde{\mathfrak{z}}^{\mathfrak{u}^0} \in PC_{1-\eta}$ as $\lim_{n \to \infty} \mathfrak{z}_n^{\mathfrak{u}} \to \tilde{\mathfrak{z}}^{\mathfrak{u}^0}$. Using Balder's theorem [49] and the property that $PC_{1-\eta} \to \mathcal{L}(\mathcal{I}, U)$ is continuous, we conclude

$$
\begin{aligned}
\inf_{\mathfrak{u} \in \mathcal{U}_{ad}} \mathfrak{I}(\mathfrak{u}) &= \lim_{n \to \infty} \int_0^T \mathfrak{L}(t, \mathfrak{z}_n^{\mathfrak{u}_n}(t), \mathfrak{u}_n(t))dt \geq \int_0^T \mathfrak{L}(t, \tilde{\mathfrak{z}}^{\mathfrak{u}^0}(t), \mathfrak{u}^0(t))dt = \mathfrak{I}(\tilde{\mathfrak{z}}^{\mathfrak{u}^0}, \mathfrak{u}^0) \\
&= \mathfrak{I}(\mathfrak{u}^0) \geq \inf_{\mathfrak{u} \in \mathcal{U}_{ad}} \mathfrak{I}(\mathfrak{u}).
\end{aligned}
$$

Therefore, $\mathfrak{I}(\mathfrak{u}) = \inf_{\mathfrak{u} \in \mathcal{U}_{ad}} \mathfrak{I}(\mathfrak{u})$, leads that $\mathcal{I}$ attains its minimum at $\mathfrak{u}^0 \in \mathcal{U}_{ad}$. Subsequently, we have

$$
\mathcal{I}(\tilde{\mathfrak{z}}^{\mathfrak{u}^0}, \mathfrak{u}^0) = \inf_{\mathfrak{u} \in \mathcal{U}_{ad}} \mathfrak{I}(\mathfrak{u}) = \inf_{(\mathfrak{z}^{\mathfrak{u}}, \mathfrak{u}) \in A_{ad}} \mathfrak{I}(\mathfrak{z}^{\mathfrak{u}}, \mathfrak{u}).
$$

Hence, $(\tilde{\mathfrak{z}}^{\mathfrak{u}^0}, \mathfrak{u}^0) \in A_{ad}$. This completes the proof.

5 Application

Consider a nonlinear equation of the form given below to validate the outcome,

$$\mathcal{D}^{\frac{1}{2},\frac{2}{3}}\left[\mathfrak{z}(\varsigma,\,\mathfrak{t}) - \exp\left(\mathfrak{z}\left(\frac{3\varsigma\mathfrak{t}}{4}\right)\right)\right] = \frac{\partial^2}{\partial \mathfrak{t}^2}\left[\mathfrak{z}(\varsigma,\,\mathfrak{t}) - \exp\left(\mathfrak{z}\left(\frac{3\varsigma\mathfrak{t}}{4}\right)\right)\right]$$

$$+ \int_0^1 \mathfrak{h}(\varsigma,\mathfrak{t})\mathfrak{u}(\varsigma,\mathfrak{t})d\mathfrak{t} + \sin\left[\exp(\varsigma\mathfrak{t}) + \mathfrak{z}\left(\frac{2\varsigma\mathfrak{t}}{5}\right)\right], \quad \varsigma \in (0,3]\setminus(1,2],$$

$$\mathfrak{z}(\varsigma,\,0) = \mathfrak{z}(\varsigma,\,\pi) = 0,\ \varsigma \in [1,2],$$ (4.1)

$$\mathfrak{z}(\varsigma,\,\mathfrak{t}) = \frac{1}{\Gamma\left(\frac{1}{2}\right)}\int_{\mathfrak{t}_k}^{\mathfrak{t}}\frac{1}{(\varsigma-\mathfrak{t})^{\frac{1}{2}}}\left(\frac{\mathfrak{z}\left(\frac{1^-}{2}\right)}{20exp(\mathfrak{t})+1}\right)d\mathfrak{t},\ \varsigma,\,\mathfrak{t} \in [1,2],$$

$$I_{0}^{\frac{5}{6}}\mathfrak{z}(0,\,\mathfrak{t}) = \int_0^3 \mathfrak{z}(\omega,\mathfrak{t})d\mathfrak{t} + 5,\ \ \mathfrak{t} \in [0,\pi],$$

with $B\mathfrak{u}(\varsigma)(\mathfrak{t}) = \int_0^1 \mathfrak{h}(\varsigma,\mathfrak{t})\mathfrak{u}(\varsigma,\mathfrak{t})d\mathfrak{t}$, & $p_1 = \frac{1}{2}$, $p_2 = \frac{2}{3}$, $\eta = \frac{5}{6}$. Assume $\mathcal{N} = L^2[0,\pi]$ and $A:$ $D(A) \subset \mathcal{N} \to \mathcal{N}$ by $A\mathfrak{z} = \frac{\partial^2}{\partial \mathfrak{t}^2}(\mathfrak{z})$,

$$D(A) = \{\mathfrak{z} \in \mathcal{N},\ \mathfrak{z}_{\mathfrak{t}}, \mathfrak{z}_{\mathfrak{t}\mathfrak{t}} \in \mathcal{N},\ \mathfrak{z}(\varsigma,0) = \mathfrak{z}(\varsigma,\pi) = 0\}.$$

It is clear that A is a strongly continuous semigroup and $(S(\varsigma)\mathfrak{z})$ in $\mathcal{N}$,

$$(S(\varsigma)\mathfrak{z})(\mathfrak{t}) = \begin{cases} \int_0^\pi \mathscr{M}(\varsigma,\mathfrak{t}-\omega)\mathfrak{z}(\omega)d\omega, & \varsigma > 0, \\ \mathfrak{z}(\mathfrak{t}), & \varsigma = 0, \end{cases}$$

with

$$\mathscr{M}(\varsigma,\mathfrak{t}) = \sqrt{\frac{2}{\pi}}exp\left(-\left(\frac{\mathfrak{t}^2}{4\varsigma}\right)\right),\ \varsigma > 0,\ 0 < \mathfrak{t} < \pi,$$

with $\mathfrak{z}(\varsigma)(\mathfrak{t}) = \mathfrak{z}(\varsigma,\mathfrak{t})$. This leads to the conclusion $||S(\varsigma)|| \leq \mathscr{M}$. Let $\mathcal{U}_{ad} = \{\mathfrak{u} \in U|||\mathfrak{u}||_{L^2[[0,3],\,U]} \leq 1\}$. Hence

$$\mathfrak{I}(\mathfrak{z},\mathfrak{u}) = \int_{\mathfrak{t}_k}^{\mathfrak{t}_{k+1}}\int_0^\pi |\mathfrak{z}(\varsigma,\mathfrak{t})|^2 d\mathfrak{t}d\varsigma + \int_{\mathfrak{t}_k}^{\mathfrak{t}_{k+1}}\int_0^\pi |\mathfrak{u}(\varsigma,\mathfrak{t})|^2 d\mathfrak{t}d\varsigma$$

related to the system (4.1) which correlates the system (1.1) with

$$\mathfrak{I}(\mathfrak{z},\mathfrak{u}) = \int_{\mathfrak{t}_k}^{\mathfrak{t}_{k+1}}\left(||\mathfrak{z}(\varsigma)||^2 + ||\mathfrak{u}(\varsigma)||_U^2\right)d\varsigma.$$

Therefore, $(H1)–(H8)$ satisfied. This completes the proof.

6 Conclusion

We examine the total controllability of non-instantaneous Hilfer fractional neutral system under integral boundary condition. By incorporating HFD with semigroup operator theory and Laplace transform technique, the integral solution is derived. Controllability outcomes were attained using Kuratowski's measure with contraction theory. Furthermore, the sequencing technique has been used to discuss the existence of the optimal pair for the system. To confirm the derived consequences, an example is given. The concept can be extended to Hilfer stochastic differential equations.

Author Contributions

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References

1. Kharade J P, Kucche K D, On the impulsive implicit Ψ–Hilfer fractional differential equations with delay, *Math. Methods Appl. Sci.*, (2019); 43(4):1938–1952. https://doi.org/10.1002/mma.6017

2. Kucche K D, Kharade J P, Sousa J V C, On the nonlinear impulsive Ψ–Hilfer fractional differential equations, *Mathematical Modelling and Analysis*, (2020); 25(4):642–660. https://doi.org/10.48550/arXiv.1901.01814

3. Hernández E, O'Regan Donal, On a new class of abstract impulsive differential equations, *Proc. Amer. Math. Soc.*, (2013); 141(5):1641–1649. https://doi.org/10.1090/S0002-9939-2012-11613-2

4. Kumar V, Stamov G, Stamova I, Controllability results for a class of piecewise nonlinear impulsive fractional dynamic systems, *Appl. Math. Comput.*, (2023); 439:127625. https://doi.org/10.1016/j.amc.2022.127625

5. Kumar V, Djemai M, Existence, stability and controllability of piecewise impulsive dynamic systems on arbitrary time domain, *Appl. Math. Model.*, (2023); 117:529–548. https://doi.org/10.1016/j.apm.2022.12.027

6. Kumar V, Kostić M, Tridane A, Debbouche A, Controllability of switched Hilfer neutral fractional dynamic systems with impulses, *IMA Journal of Mathematical Control and Information*, (2022); 39(3):807–836. https://doi.org/10.1093/imamci/dnac011

7. Malik M, Sajid M, Kumar V, Controllability of singular dynamic systems on time scales, *Asian Journal of Control*, (2021); 24(5):2771–2777. https://doi.org/10.1002/asjc.2645

8. Chen P, Zhang X, Li Y, Existence of mild solutions to partial differential equations with non-instantaneous impulses, *Electron. J. Differ. Equ.*, (2016); 241:1–11.

9. Saravanakumar S, Balasubramaniam P, Non-instantaneous impulsive Hilfer fractional stochastic differential equations driven by fractional Brownian motion, *Stoch. Anal. Appl.*, (2021); 39(3):549–566. https://doi.org/10.1080/07362994.2020.1815545

10. Wang J R, Ibrahim G, O'Regan Donal, Zhou Y, A general class of non-instantaneous fractional differential inclusions in Banach spaces, *Adv. Difference Equ.*, (2017); 287(2017). https://doi.org/10.1186/s13662-017-1342-8

11. Wang J R, Ibrahim G, O'Regan Donal, Controllability of Hilfer fractional non-instantaneous impulsive semilinear differential inclusions with nonlocal conditions, *Nonlinear Anal. Model. Control*, (2019); 24 (6):958–984. https://doi.org/10.15388/NA.2019.6.7

12. Liu S, Wang J R, O'Regan Donal, Trajectory approximately controllability and optimal control for noninstantaneous impulsive inclusions without compactness, *Topol. Methods. Nonlinear. Anal.*, (2021); 58 (1):19–49. https://doi.org/10.12775/TMNA.2020.069

13. Alsheekhhussain Z, Gamal Ibrahim A, Controllability of Semilinear Multi-Valued Differential Inclusions with Non-Instantaneous Impulses of Order $\alpha \in (1, 2)$ without Compactness, *Symmetry*, (2021); 13 (4):566. https://doi.org/10.3390/sym13040566

14. Ramkumar K, Ravikumar K, Baleanu D, Anguraj A, Hilfer fractional neutral stochastic differential equations with non-instantaneous impulses, *AIMS Mathematics*, (2021); 6(5):4474–4491. https://doi.org/10.3934/math.2021265

15. Agarwal R, Almeida R, Hristova S, O'Regan Donal, Non-instantaneous impulsive fractional differential equations with state dependent delay and practical stability, *Acta Mathematica Scientia*, (2021); 41:1699–1718. https://doi.org/10.1007/s10473-021-0518-1

16. Borah J, Bora S N, Non-instantaneous impulsive fractional semilinear evolution equations with finite delay, *J. Fractional Calc. & Appl.*, (2021); 12(1):120–132. https://doi.org/10.21608/jfca.2021.308746

17. Zhou Y, Jiao F, Nonlocal Cauchy problem for fractional evolution equations, *Nonlinear Anal. Real World Applications*, (2010); 11(5):4465–4475. https://doi.org/10.1016/j.nonrwa.2010.05.029

18. Wang X, Wang L, Zeng Q, Fractional differential equations with integral boundary conditions, *J. Nonlinear Sci. Appl.*, (2015); 8(4):309–314. https://doi.org/10.22436/jnsa.008.04.03

19. Wang J R, Zhang Y, Nonlocal initial value problems for differential equations with Hilfer fractional derivative, *Appl. Math. Comput.*, (2015); 266:850–859. https://doi.org/10.1016/j.amc.2015.05.144

20. Hilfer R, *Applications of Fractional Calculus in Physics*, World Scientific, Singapore, (2000).

21. Furati K M, Kassim M D, Tatar N E, Existence and uniqueness for a problem involving Hilfer fractional derivative, *Comput. Math. Appl.*, (2012); 64(6):1616–1626. https://doi.org/10.1016/j.camwa.2012.01.009

22. Gu H, Trujillo J J, Existence of mild solution for evolution equation with Hilfer fractional derivative, *Appl. Math.Comput.*, (2015); 257:344–354. https://doi.org/10.1016/j.amc.2014.10.083

23. Hilfer R, Luchko Y, Tomovski Z, Operational method for the solution of fractional differential equations with generalized Riemann-Liouville fractional derivatives, *Fract. Calc. Appl. Anal.*, (2009); 12(3):299–318. https://doi.org/10.1515/fca-2021-0023

24. Subashini R, Jothimani K, Nisar K S, Ravichandran C, New results on nonlocal functional integro-differential equations via Hilfer fractional derivative, *Alexandria Engineering Journal*, (2020); 59(5):2891–2899. https://doi.org/10.1016/j.aej.2020.01.055

25. Agarwal P, Baleanu D, Quan Y, Momani C S, Machado J A, *Fractional Calculus- Models*, *Algorithms*, *Technology*, Springer, Singapore (2018).

26. Kilbas A A, Srivastava H M, Trujillo J J, *Theory and applications of fractional differential equations* In: North-Holland Mathematics Studies, Elsevier Science, Amsterdam; (2006):204.

27. Pazy A, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, New York, Springer-verlag (1983).

28. Podlubny I, *Fractional Differential Equations. An introduction to fractional derivatives*, *fractional differential equations*, *to methods of their solution and some of their applications*, Academic Press, San Diego (1999).

29. Zhou Y, *Basic Theory of Fractional Differential Equations*, World Scientific, Singapore, (2014).

30. Ravichandran C, Jothimani K, Nisar K S, Mahmoud Emad E, Yahia Ibrahim S, An interpretation on controllability of Hilfer fractional derivative with nondense domain, *Alexandria Engineering Journal*, (2022); 61(12):9941–9948. https://doi.org/10.1016/j.aej.2022.03.011

31. Nisar K S, Jothimani K, Kaliraj K, Ravichandran C, An analysis of controllability results for nonlinear Hilfer neutral fractional derivatives with non-dense domain, *Chaos Solitons & Fractals*, (2021); 146:110915. https://doi.org/10.1016/j.chaos.2021.110915

32. Nisar K S, Jothimani K, Ravichandran C, Baleanu D, Kumar D, New approach on controllability of Hilfer fractional derivatives with nondense domain, *AIMS Mathematics*, (2022); 7(6):10079–10095. https://doi.org/10.3934/math.2022561

33. Nisar K S, Jagatheeshwari R, Ravichandran C, Veeresha P, High performance computational method for fractional model of solid tumour invasion, *Ain Shams Eng. J.*, https://doi.org/10.1016/j.asej.2023.102226

34. Jothimani K, Kaliraj K, Panda S K, Nisar K S, Ravichandran C, Results On controllability of non-densely characterized neutral fractional delay differential system, *Evol. Equ. Control Theory*, 2021; 10(3):619–631. https://doi.org/10.3934/eect.2020083

35. Goufo E F D, Ravichandran C, Birajdar G A, Self-similarity techniques for chaotic attractors with many scrolls using step series switching, *Math. Model. Anal.*, (2021); 26(4):591–611. https://doi.org/10.3846/mma.2021.13678

36. Kavitha K, Vijayakumar V, Udhayakumar R, Results on controllability of Hilfer fractional neutral differential equations with infinite delay via measures of noncompactness, *Chaos Solitons & Fractals*, (2020); 139:1–9. https://doi.org/10.1016/j.chaos.2020.110035

37. Kavitha K, Vijayakumar V, Udhayakumar R, Ravichandran C, Results on controllability of Hilfer fractional differential equations with infinite delay via measures of noncompactness, *Asian J. Control*, (2022); 24(3):1406–1415. https://doi.org/10.1002/asjc.2549

38. Du J, Jiang W, Pang D, Niazi A U K, Exact controllability for Hilfer fractional differential inclusions involving nonlocal initial conditions, *Complexity*, (2018); 2018:1–13. https://doi.org/10.1155/2018/9472847

39. Lv J, Yang X, Approximate controllability of Hilfer fractional differential equations, *Math. Methods Appl. Sci.*, (2020); 43(1):242–254. https://doi.org/10.1002/mma.5862

40. Wang J R, Liu X, O'Regan Donal, On the approximate controllability for Hilfer fractional evolution hemivariational inequalities, *Numer. Funct. Anal. Optim.*, (2019); 40 (7):743–762. https://doi.org/10.1080/01630563.2018.1499667

41. Bahaa G M, Optimal control problem and maximum principle for fractional order cooperative systems, *Kybernetika*, (2019); 55(2):337–358. https://doi.org/10.14736/kyb-2019-2-0337

42. Harrat A, Nieto J J, Debbouche A, Solvability and optimal controls of impulsive Hilfer fractional delay evolution inclusions with Clarke subdifferential, *J. Comput. Appl. Math.*, (2018); 344:725–737. https://doi.org/10.1016/j.cam.2018.05.031

43. Pan X, Li X, Zhao J, Solvability and optimal controls of semi linear Riemann- Liouville fractional differential equations, *Abstr. Appl. Anal.*, (2014);(2014):216919. https://doi.org/10.1155/2014/216919

44. Sousa J V C, Existence of mild solutions to Hilfer fractional evolution equations in Banach space, *Classical Analysis and ODEs*, 1812.02213v1

45. Kumar V, Malik M, Debbouche A, Total controllability of neutral fractional differential equation with non-instantaneous impulsive effects, *J. Comput. Appl. Math.*, (2021); 383:113158. https://doi.org/10.1016/j.cam.2020.113158

46. Yang H, Zhao Y, Existence and optimal controls of non-autonomous impulsive integro-differential evolution equation with nonlocal conditions, *Chaos Solitons & Fractals*, (2021); 148:111027. https://doi.org/10.1016/j.chaos.2021.111027

47. Banaś J, On measures of noncompactness in Banach spaces, *Comment. Math. Univ. Carolin.*, (1980); 21(1):131–143.

48. Zhu S, Fan Z, Li G, Optimal controls for Riemann-Liouville fractional evolution systems without Lipschitz assumption, *J. Optim. Theory. Appl.*, (2017); 174(1):47–64. https://doi.org/10.1007/s10957-017-1119-y

49. Balder E, Necessary and sufficient conditions for l1-strong-weak lower semi-continuity of integral functional, *Nonlinear Anal.*, (1987); 11(12):1399–1404. https://doi.org/10.1016/0362-546X(87)90092-7