RESEARCH ARTICLE

Use of an efficient unbiased estimator for finite population mean

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Abstract

In this study, we propose an improved unbiased estimator in estimating the finite population mean using a single auxiliary variable and rank of the auxiliary variable by adopting the Hartley-Ross procedure when some parameters of the auxiliary variable are known. Expressions for the bias and mean square error or variance of the estimators are obtained up to the first order of approximation. Four real data sets are used to observe the performances of the estimators and to support the theoretical findings. It turns out that the proposed unbiased estimator outperforms as compared to all other considered estimators. It is also observed that using conventional measures have significant contributions in achieving the efficiency of the estimators.

1. Introduction

In literature, many researchers have constructed or modified several forms of ratio, product, and regression type estimators by using the auxiliary information in estimating the finite population mean. The auxiliary information can be used either at survey stage or designing stage or estimation stage or at all stages to enhance the precision of the estimators by taking the advantage of correlation between the study variable and the auxiliary variable. In this study, we use the auxiliary variable as well as rank of the auxiliary variable at estimation stage to estimate the finite population mean. [1] were the pioneer whom used the idea of ratio of the study variable and the auxiliary variable in estimating the population mean. Singh and Singh [2] suggested the [1] type estimator when some parameters of the auxiliary variable are known in advance. [3] slightly modified the idea of [1] and suggested a new estimator for estimating the population mean. [4] used the known population parameters of the auxiliary variable in their suggested estimator for mean estimation. [5] extended the [1] estimator by using two auxiliary variables to estimate the population mean. [6, 7] modified the [1] type estimator for mean estimation in simple and stratified sampling. [8] have given justification in their proposed estimator by using dual use of the auxiliary variable in their study. [9] used the dual auxiliary variable in estimating the mean of the sensitive variable under randomized response technique (RRT). [10] modified the existing ratio estimator by using the dual auxiliary information for mean estimation. [11] suggested a difference type exponential estimator based on dual auxiliary

variable for mean estimation. Recently [12] suggested a difference type estimator using the dual auxiliary variable under non-response in simple random sampling.

There are several estimators exist in literature which give the biased results and consequently variance or MSE tend to be inflated. This serious drawback encouraged us to construct the unbiased estimator which should be better than other considered estimators in literature. So combining the ideas of [1] and [2], we suggest an improved unbiased estimator for estimating the finite population mean.

In Section 2, we introduce some useful notations and symbols. Section 3 gives the existing estimators in literature. The proposed estimator is discussed in Section 4. The numerical results based on four real data sets are mentioned in Section 5. The conclusion is given in Section 6.

2. Symbols and notations

Consider a finite population $\Lambda = \Lambda\{\Lambda_1, \Lambda_2, \ldots, \Lambda_N\}$ of N units. A simple random sample without replacement (SRSWOR) is used to draw a sample of size n units from a population. Let y_i, x_i, and r_i be the observed values of the study variable (Y), the auxiliary variable (X) and rank of the auxiliary variable (R) respectively. Let $\bar{y} = \sum_{i=1}^{n} y_i/n$, $\bar{x} = \sum_{i=1}^{n} x_i/n$ and $\bar{r} = \sum_{i=1}^{n} r_i/n$ respectively be the sample means corresponding to the population means $\bar{Y} = \sum_{i=1}^{N} y_i/N$, $\bar{X} = \sum_{i=1}^{N} x_i/N$ and $\bar{R} = \sum_{i=1}^{N} r_i/N$. Let $s_y^2 = \sum_{i=1}^{n} (y_i - \bar{y})^2/(n-1)$, $s_x^2 = \sum_{i=1}^{n} (x_i - \bar{x})^2/(n-1)$, and $s_r^2 = \sum_{i=1}^{N} (r_i - \bar{r})^2/(n-1)$ respectively be the sample variances corresponding to population variances $S_y^2 = \sum_{i=1}^{N} (y_i - \bar{Y})^2/(N-1)$, $S_x^2 = \sum_{i=1}^{N} (x_i - \bar{X})^2/(N-1)$, and $S_r^2 = \sum_{i=1}^{N} (r_i - \bar{R})^2/(N-1)$. Let $C_y = S_y/\bar{Y}$, $C_x = S_x/\bar{X}$, and $C_r = S_r/\bar{R}$ be the coefficients of variation of Y, X, and R respectively. Let $\rho_{yx} = S_{yx}/(S_y S_x)$, $\rho_{yr} = S_{yr}/(S_y S_r)$, and $\rho_{xr} = S_{xr}/(S_x S_r)$, be the correlation coefficients between their respective subscripts, where $S_{yx} = \sum_{i=1}^{N} (y_i - \bar{Y})(x_i - \bar{X})/(N-1)$, $S_{yr} = \sum_{i=1}^{N} (y_i - \bar{Y})(r_i - \bar{R})/(N-1)$, and $S_{xr} = \sum_{i=1}^{N} (x_i - \bar{X})(r_i - \bar{R})/(N-1)$ be the population covariances between their respective subscripts. Corresponding sample covariances are $s_{yx} = \sum_{i=1}^{n} (y_i - \bar{y})(x_i - \bar{x})/(n-1)$, $s_{yr} = \sum_{i=1}^{n} (y_i - \bar{y})(r_i - \bar{r})/(n-1)$, and $s_{xr} = \sum_{i=1}^{n} (x_i - \bar{x})(r_i - \bar{r})/(n-1)$.

We define the following relative error terms to derive bias and MSE or variance expressions.

Let $\Psi_0 = \frac{\bar{y}}{\bar{Y}} - 1$ and $\Psi_1 = \frac{\bar{x}}{\bar{X}} - 1$, $\Psi_2 = \frac{\bar{r}}{\bar{R}} - 1$, $\Psi_3 = \frac{s_{yx}}{S_{yx}} - 1$, such that $E(\Psi_i) = 0$, $(i = 0,1,2,3)$, $E(\Psi_0^2) = \Upsilon C_y^2$, $E(\Psi_1^2) = \Upsilon C_x^2$, $E(\Psi_2^2) = \Upsilon C_r^2$, $E(\Psi_3^2) = \Upsilon\left(\frac{\Delta_{220}}{\rho_{yx}} - 1\right)$, $E(\Psi_0\Psi_1) = \Upsilon C_{yx}$, $E(\Psi_0\Psi_2) = \Upsilon C_{yr}$, $E(\Psi_0\Psi_3) = \Upsilon C_y \frac{\Delta_{210}}{\rho_{yx}}$, $E(\Psi_1\Psi_2) = \Upsilon C_{xr}$, $E(\Psi_1\Psi_3) = \Upsilon C_{yx}\frac{\Delta_{120}}{\rho_{yx}}$, $E(\Psi_2\Psi_3) = \Upsilon C_r \frac{\Delta_{102}}{\rho_{yr}}$. where $C_{yx} = \rho_{yx}C_yC_x$, $C_{yr} = \rho_{yr}C_yC_r$, $C_{xr} = \rho_{xr}C_xC_r$, $\Delta_{abc} = \frac{\mu_{abc}}{\mu_{200}^{a/2}\mu_{020}^{b/2}\mu_{002}^{c/2}}$, $\mu_{abc} = \frac{1}{(N-1)}\sum_{i=1}^{N} (y_i - \bar{Y})^a (x_i - \bar{X})^b (r_i - \bar{R})^c$, and $\Upsilon = \left(\frac{1}{n} - \frac{1}{N}\right)$.

3. Existing estimators

Now we discuss some well-known estimators in estimating the finite population mean.

1. The usual sample mean estimator is $\bar{y}_{(0)} = \bar{y}$, and its variance, is given by

$$Var(\bar{y}_{(0)}) = \Upsilon \bar{Y}^2 C_y^2. \tag{1}$$

2. A general class of Hartley-Ross unbiased type estimators, is given by

$$\bar{y}_{(G)}^{(U)} = \bar{k}^{(j)} \bar{X}^{(j)} + \frac{n(N-1)}{N(n-1)} \left(\bar{y} - \bar{k}^{(j)} \bar{x}^{(j)} \right), \tag{2}$$

where $\bar{x}^{(j)} = c\bar{x} + d$, $\bar{X}^{(j)} = c\bar{X} + d$, $\bar{k}^{(j)} = \frac{1}{n}\sum_{i=1}^{n} k_i^{(j)}$, $k_i^{(j)} = \frac{y_i}{x_i^{(j)}}$, $x_i^{(j)} = cx_i + d$, $E(y_i) = \bar{Y}$,

$E(k_i^{(j)}) = \bar{K}^{(j)}$, $\bar{K}^{(j)} = \frac{1}{N}\sum_{i=1}^{N} k_i^{(j)}$; $j = 0, 1, 2$; c and d are the known population parameters of the auxiliary variable which may be coefficient of variation (C_x), coefficient of skewness (β_{1x}), coefficient of kurtosis (β_{2x}) and correlation coefficient (ρ_{yx}).

Using the assumption $\frac{n(N-1)}{N(n-1)} \approx 1$ and $(\bar{Y}^{(j)} - \bar{K}^{(j)}\bar{X}^{(j)}) \approx 0$, an unbiased general estimator is given by

$$\bar{y}_{(G)}^{(U)} \approx \bar{y} + \bar{k}^{(j)} \left(\bar{X}^{(j)} - \bar{x}^{(j)} \right). \tag{3}$$

The variance of $\bar{y}_{(G)}^{(U)}$, is given by

$$Var(\bar{y}_{(G)}^{(U)}) \approx \Upsilon \left[S_y^2 + \bar{K}^{(j)2} S_{x^{(j)}}^2 - 2\bar{K}^{(j)} S_{yx^{(j)}} \right], \tag{4}$$

where $S_{x^{(j)}}^2 = \frac{1}{(N-1)}\sum_{i=1}^{N} (x_i^{(j)} - \bar{X}^{(j)})^2$, and $S_{yx^{(j)}} = \frac{1}{(N-1)}\sum_{i=1}^{N}(y_i - \bar{Y})(x_i^{(j)} - \bar{X}^{(j)}) = \rho_{yx^{(j)}} S_y S_{x^{(j)}}$.

Note:

1. Put $c = 1$, $d = 0$, in (3), so $j = 0$, i.e. $\bar{x}^{(0)} = \bar{x}$, $\bar{X}^{(0)} = \bar{X}$, $\bar{k}^{(0)} = \frac{1}{n}\sum_{i=1}^{n} k_i^{(0)}$, and $\bar{K}^{(0)} = \frac{1}{N}\sum_{i=1}^{N} k_i^{(0)}$, where $k_i^{(0)} = \frac{y_i}{x_i^{(0)}} = \frac{y_i}{x_i}$, we get the usual Hartley-Ross estimator and its variance as:

$$\bar{y}_{(HR)}^{(U)} \approx \bar{y} + \bar{k}^{(0)} \left(\bar{X}^{(0)} - \bar{x}^{(0)} \right), \tag{5}$$

and

$$Var(\bar{y}_{(HR)}^{(U)}) \approx \Upsilon \left[S_y^2 + \bar{K}^{(0)2} S_{x^{(0)}}^2 - 2\bar{K}^{(0)} S_{yx^{(0)}} \right]. \tag{6}$$

2. Put $c = C_x$, $d = B_{2(x)}$, in (3), so $j = 1$, i.e. $\bar{x}^{(1)} = C_x\bar{x} + \beta_{2(x)}$, $\bar{X}^{(1)} = C_x\bar{X} + \beta_{2(x)}$, $\bar{k}^{(1)} = \frac{1}{n}\sum_{i=1}^{n} k_i^{(1)}$, and $\bar{K}^{(1)} = \frac{1}{N}\sum_{i=1}^{N} k_i^{(1)}$, where $k_i^{(1)} = \frac{y_i}{x_i^{(1)}} = \frac{y_i}{C_x x_i + \beta_{2(x)}}$, we get the [4] estimator with its variance, are given by:

$$\bar{y}_{(S_1)}^{(U)} \approx \bar{y} + \bar{k}^{(1)} \left(\bar{X}^{(1)} - \bar{x}^{(1)} \right), \tag{7}$$

and

$$Var(\bar{y}_{(S_1)}^{(U)}) \approx \Upsilon \left[S_y^2 + \bar{K}^{(1)2} S_{x^{(1)}}^2 - 2\bar{K}^{(1)} S_{yx^{(1)}} \right]. \tag{8}$$

3. Put $c = C_x$, $d = \rho_{yx}$, in (3), so $j = 2$, i.e. $\bar{x}^{(2)} = C_x\bar{x} + \rho_{yx}$, $\bar{X}^{(2)} = C_x\bar{X} + \rho_{yx}$, $\bar{k}^{(2)} = \frac{1}{n}\sum_{i=1}^{n}k_i^{(2)}$,

and $\bar{K}^{(2)} = \frac{1}{N}\sum_{i=1}^{N}k_i^{(2)}$, where $k_i^{(2)} = \frac{y_i}{x_i^{(2)}} = \frac{y_i}{C_x x_i + \rho_{yx}}$, we get another [4] estimator and its variance, is given by:

$$\bar{y}_{(S2)}^{(U)} \approx \bar{y} + \bar{k}^{(2)}\left(\bar{X}^{(2)} - \bar{x}^{(2)}\right), \tag{9}$$

and

$$Var(\bar{y}_{(S_2)}^{(U)}) \approx \Upsilon\left[S_y^2 + \bar{K}^{(2)2}S_{x^{(2)}}^2 - 2\bar{K}^{(2)}S_{yx^{(2)}}\right]. \tag{10}$$

4. A difference type estimator using a single auxiliary variable with its ranks, is given by:

$$\bar{y}_{(D)}^{(U)} = [\bar{y} + d_1(\bar{X} - \bar{x}) + d_2(\bar{R} - \bar{r})], \tag{11}$$

where $d_i(i = 1, 2)$ are constants.

The variance of $\bar{y}_{(D)}^{(U)}$, is given by

$$Var(\bar{y}_{(D)}^{(U)}) = \left\{\gamma\bar{Y}^2C_y^2 + d_1^2\bar{X}^2m_1 + d_2^2\bar{R}^2m_2 - 2d_1\bar{Y}\bar{X}m_3 - 2d_2\bar{Y}\bar{R}m_4 + 2d_1d_2\bar{X}\bar{R}m_5\right\},$$

where

$$m_1 = \Upsilon C_x^2, m_2 = \Upsilon C_r^2, m_3 = \Upsilon C_{yx}, m_4 = \Upsilon C_{yr}, m_5 = \Upsilon C_{xr}.$$

The minimum variance of $\bar{y}_{(D)}^{(U)}$, at optimum values of $d_i(i = 1, 2)$ i.e. $d_{1(opt)} = \frac{\bar{Y}}{\bar{X}}\left(\frac{m_2m_3-m_4m_5}{m_1m_2-m_5^2}\right)$ and $d_{2(opt)} = \frac{\bar{Y}}{\bar{R}}\left(\frac{m_1m_4-m_3m_5}{m_1m_2-m_5^2}\right)$, is given by

$$Var(\bar{y}_{(D)}^{(U)})_{\min} \approx \bar{Y}^2\left[\Upsilon C_y^2 - \frac{\Delta_1}{(m_1m_2 - m_5^2)}\right], \tag{12}$$

where

$$\Delta_1 = m_1m_4^2 + m_2m_3^2 - 2m_3m_4m_5.$$

5. [6] suggested the following unbiased estimator using the single auxiliary variable and is given by:

$$\bar{y}_{(CK)}^{(U)} = Q\bar{y}\left(\frac{c\bar{X} + d}{\alpha(c\bar{x} + d) + (1 - \alpha)(c\bar{X} + d)}\right)^t - (Q - 1)\bar{y} - Q\Upsilon\bar{y}\left[\frac{t(t+1)}{2}\Upsilon^2\alpha^2C_x^2 - \alpha\Upsilon t\frac{s_{yx}}{\bar{y}\bar{X}}\right],$$

where c and d are defined earlier i.e. $t = -1, 0, 1$; $\alpha = 0, 1$ and Q is a constant whose value is to be estimated.

For $\alpha = t = 1$, the above estimator becomes:

$$\bar{y}_{(CK)}^{(U)} = Q\bar{y}\left(\frac{c\bar{X} + d}{c\bar{x} + d}\right) - (Q - 1)\bar{y} - Q\Upsilon\bar{y}\left[g^2C_x^2 - g\frac{s_{yx}}{\bar{y}\bar{X}}\right], \tag{13}$$

$g = \frac{c\bar{X}}{c\bar{X}+d}$.

The minimum variance of $\bar{y}_{(CK)}^{(U)}$ at optimum values of Q i.e. $Q_{opt} = -\frac{\nabla_2}{\nabla_1}$, is given by

$$Var(\bar{y}_{(CK)}^{(U)})_{\min} \approx \bar{Y}^2\left[\Upsilon C_y^2 - \frac{\nabla_2^2}{\nabla_1}\right], \tag{14}$$

where $\nabla_1 = \nabla_{1a} + \nabla_{1b}$,

$$\nabla_{1a} = \Upsilon^3 g^4 C_x^4 C_y^2 + \Upsilon g^2 C_x^2 + \Upsilon^2 C_{yx} g C_y \frac{\Delta_{210}}{\rho_{yx}} - \Upsilon^2 g^4 C_x^4 - \Upsilon^2 g^2 C_{yx}^2 + \Upsilon^3 C_{yx}^2 g^2 \left(\frac{\Delta_{220}}{\rho_{yx}} - 1 \right),$$

$$\nabla_{1b} = 4\Upsilon^2 g^3 C_{yx} C_x^2 - 2\Upsilon^2 g^2 C_{yx} C_x \frac{\Delta_{120}}{\rho_{yx}} - 2\Upsilon^3 g^3 C_{yx} C_x^2 C_y \frac{\Delta_{210}}{\rho_{yx}},$$

$$\nabla_2 = -\Upsilon g C_{yx} - \Upsilon^2 g^2 C_x^2 C_y^2 + \Upsilon^2 g C_{yx} C_y \frac{\Delta_{210}}{\rho_{yx}}.$$

6. [8] suggested an idea of using rank of the auxiliary variable in the following estimator, is given by

$$\bar{y}_{(H)} = [H_1 \bar{y} + H_2(\bar{X} - \bar{x}) + H_3(\bar{R} - \bar{r})] \exp\left(\frac{c(\bar{X} - \bar{x})}{c(\bar{X} + \bar{x}) + 2d} \right), \tag{15}$$

where $H_i(i = 1,2,3)$ are constants; c and d are defined earlier.

The bias of the estimator $\bar{y}_{(H)}$, is given by

$$B(\bar{y}_{(H)}) \approx (H_1 - 1)\bar{Y} + H_1 \bar{Y} \Upsilon \left(\frac{3}{8} g^2 C_x^2 - \frac{1}{2} g C_{yx} \right) + H_2 \bar{X} \frac{1}{2} g \Upsilon C_x^2 + H_3 \bar{R} \frac{1}{2} g \Upsilon C_{xr}. \tag{16}$$

Since $\bar{X}$ and $\bar{R}$ are known, so replacing $\bar{Y}$ and C_{yx} by their consistent estimates $\bar{y}$ and $\hat{C}_{yx} = \frac{s_{yx}}{\bar{y}\bar{X}}$ in (16), the estimated bias of $\bar{y}_{(H)}$ becomes

$$\hat{B}(\bar{y}_{(H)}) \approx (H_1 - 1)\bar{y} + H_1 \bar{y} \Upsilon \left(\frac{3}{8} g^2 C_x^2 - \frac{1}{2} g \frac{s_{yx}}{\bar{y}\bar{X}} \right) + H_2 \bar{X} \frac{1}{2} g \Upsilon C_x^2 + H_3 \bar{R} \frac{1}{2} g \Upsilon C_{xr}. \tag{17}$$

Subtracting $\hat{B}(\bar{y}_{(H)})$ from $\bar{y}_{(H)}$, we get an unbiased estimator by replacing $H_i(i = 1,2,3)$ by $L_i(i = 1,2,3)$ which is considered by [10] as:

$$\bar{y}_{(I)}^{(U)} = \bar{y}_{(H)} - \hat{B}(\bar{y}_{(H)}) \tag{18}$$

or

$$\begin{aligned}\bar{y}_{(I)}^{(U)} = \quad & [L_1 \bar{y} + L_2(\bar{X} - \bar{x}) + L_3(\bar{R} - \bar{r})] \exp\left(\frac{c(\bar{X} - \bar{x})}{c(\bar{X} + \bar{x}) + 2d} \right) - (L_1 - 1)\bar{y} \\ & - L_1 \bar{y} \Upsilon \left(\frac{3}{8} g^2 C_x^2 - \frac{1}{2} g \frac{s_{yx}}{\bar{y}\bar{X}} \right) - L_2 \bar{X} \frac{1}{2} g \Upsilon C_x^2 - L_3 \bar{R} \frac{1}{2} g \Upsilon C_{xr}. \end{aligned} \tag{19}$$

Rewriting in terms of errors, we have

$$\begin{aligned}\bar{y}_{(I)}^{(U)} - \bar{Y} \approx \ & \bar{Y}\Psi_0 + L_1 \bar{Y} \left[-\frac{3}{8}\Upsilon g^2 C_x^2 \Psi_0 - \frac{1}{2}g\Psi_1 + \frac{1}{2}\Upsilon C_{yx} g \Psi_3 - \frac{1}{2}g\Psi_0\Psi_1 + \frac{3}{8}g^2\Psi_1^2 \right. \\ & \left. - \frac{3}{8}g^2 \Upsilon C_x^2 + \frac{1}{2}\Upsilon g C_{yx} \right] - L_2 \bar{X} \left[\Psi_1 - \frac{1}{2}g\Psi_1^2 + \frac{1}{2}g\Upsilon C_x^2 \right] \\ & - L_3 \bar{R} \left[\Psi_2 - \frac{1}{2}g\Psi_1\Psi_2 + \frac{1}{2}g\Upsilon C_{xr} \right] \end{aligned} \tag{20}$$

Solving (20), the bias of $\bar{y}_{(I)}^{(U)}$ becomes zero. Now squaring and taking expectation of (20), the variance of $\bar{y}_{(I)}^{(U)}$ becomes:

$$Var(\bar{y}_{(I)}^{(U)}) = \ \Upsilon\bar{Y}^2 C_y^2 + L_1^2\bar{Y}^2 h_1 + L_2^2\bar{X}^2 h_2 + L_3^2\bar{R}^2 h_3 + 2L_1\bar{Y}^2 h_4 - 2L_2\bar{Y}\bar{X}h_5$$
$$-2L_3\bar{Y}\bar{R}h_6 - 2L_1L_2\bar{Y}\bar{X}h_7 - 2L_1L_3\bar{Y}\bar{R}h_8 + 2L_2L_3\bar{X}\bar{R}h_9, \tag{21}$$

where

$$h_1 = \ \frac{1}{4}\Upsilon g^2 C_x^2 - \frac{9}{64}\Upsilon^2 g^4 C_x^4 - \frac{1}{4}\Upsilon^2 g^2 C_{yx}^2 - \frac{1}{2}\Upsilon^2 g^2 C_{yx}C_x\frac{\Delta_{120}}{\rho_{yx}} + \frac{3}{4}\Upsilon^2 g^3 C_{yx}C_x^2$$
$$+ \frac{9}{64}\Upsilon^3 g^4 C_y^2 C_x^4 + \frac{1}{4}\Upsilon^3 g^2 C_{yx}^2\left(\frac{\Delta_{220}}{\rho_{yx}} - 1\right) - \frac{3}{8}\Upsilon^3 g^3 C_x^2 C_{yx}C_y\left(\frac{\Delta_{210}}{\rho_{yx}}\right),$$

$$h_2 = \Upsilon C_x^2 - \frac{1}{4}\Upsilon^2 g^2 C_x^4,$$

$$h_3 = \Upsilon C_r^2 - \frac{1}{4}\Upsilon^2 g^2 C_{xr}^2,$$

$$h_4 = -\frac{3}{8}\Upsilon^2 g^2 C_y^2 C_x^2 + \frac{1}{2}\Upsilon^2 g C_{yx}C_y\frac{\Delta_{210}}{\rho_{yx}} - \frac{1}{2}\Upsilon g C_{yx},$$

$$h_5 = \Upsilon C_{yx},$$

$$h_6 = \Upsilon C_{yr},$$

$$h_7 = -\frac{5}{8}\Upsilon^2 g^2 C_{yx}C_x^2 - \frac{1}{2}\Upsilon g C_x^2 + \frac{1}{2}\Upsilon^2 g C_{yx}C_x\frac{\Delta_{120}}{\rho_{yx}} + \frac{3}{16}\Upsilon^2 g^3 C_x^4,$$

$$h_8 = -\frac{3}{8}\Upsilon^2 g^2 C_x^2 C_{yr} - \frac{1}{2}\Upsilon g C_{xr} + \frac{1}{2}\Upsilon^2 g C_{yx}C_r\frac{\Delta_{102}}{\rho_{yr}} + \frac{3}{16}\Upsilon^2 g^3 C_x^2 C_{rz} - \frac{1}{4}\Upsilon^2 g^2 C_{yx}C_{xr},$$

$$h_9 = \Upsilon C_{xr} - \frac{1}{4}\Upsilon^2 g^2 C_x^2 C_{xr}.$$

The optimum values $L_i(i = 1,2,3)$ are $L_{1(opt)} = -\frac{T_2}{T_1}, L_{2(opt)} = \frac{\bar{Y}}{\bar{X}}\left(\frac{T_3}{T_1}\right), L_{3(opt)} = \frac{\bar{Y}}{\bar{R}}\left(\frac{T_4}{T_1}\right).$

The minimum variance of $\bar{y}_{(I)}^{(U)}$ is given by

$$Var(\bar{y}_{(I)}^{(U)})_{\min} \approx \bar{Y}^2\left[\Upsilon C_y^2 - \frac{T_5}{T_1}\right], \tag{22}$$

where

$$T_1 = h_1 h_2 h_3 - h_1 h_9^2 - h_2 h_8^2 - h_3 h_7^2 + 2h_7 h_8 h_9,$$

$$T_2 = h_2 h_3 h_4 - h_2 h_6 h_8 - h_3 h_5 h_7 - h_4 h_9^2 + h_5 h_8 h_9 + h_6 h_7 h_8,$$

$$T_3 = h_1 h_3 h_5 - h_1 h_6 h_9 - h_3 h_4 h_7 + h_4 h_8 h_9 - h_5 h_8^2 + h_6 h_7 h_8,$$

$$T_4 = h_1 h_2 h_6 - h_1 h_5 h_9 - h_2 h_4 h_8 + h_4 h_7 h_9 + h_5 h_7 h_8 - h_6 h_7^2,$$

$$T_5 = T_{5a} + T_{5b},$$

$$T_{5a} = h_1 h_2 h_6^2 + h_2 h_3 h_4^2 - 2h_2 h_4 h_6 h_8 + h_1 h_3 h_5^2 - 2h_1 h_5 h_6 h_9 - 2h_3 h_4 h_5 h_7,$$

$$T_{5b} = -h_4^2 h_9^2 + 2h_4 h_5 h_8 h_9 + 2h_4 h_6 h_7 h_9 - h_5^2 h_8^2 + 2h_5 h_6 h_7 h_8 - h_6^2 h_7^2.$$

4. Proposed almost unbiased estimator

On the lines of [1, 8, 10], we propose the following alternative new unbiased estimator. This estimator is based on usual ratio, difference, and exponential ratio type estimators. The purpose is to construct an unbiased estimator that should be better than all considered estimators in estimating the finite population mean.

$$\bar{y}_{(P)} = \left[S_1 \bar{y} \left(\frac{c\bar{X} + d}{c\bar{x} + d} \right) + S_2(\bar{X} - \bar{x}) + S_3(\bar{R} - \bar{r}) \right] \exp \left(\frac{c(\bar{X} - \bar{x})}{c(\bar{X} + \bar{x}) + 2d} \right), \tag{23}$$

where $S_i (i = 1,2,3)$ are constants.

Rewriting (23) in terms of errors, we have

$$\begin{aligned} \bar{y}_{(P)} - \bar{Y} = \quad & (S_1 - 1)\bar{Y} + S_1\bar{Y} \left[\Psi_0 - \frac{3}{2} g\Psi_1 - \frac{3}{2} g\Psi_0\Psi_1 + \frac{15}{8} g^2\Psi_1^2 \right] \\ & - S_2\bar{X} \left[\Psi_1 - \frac{1}{2} g\Psi_1^2 \right] - S_3\bar{R} \left[\Psi_2 - \frac{1}{2} g\Psi_1\Psi_2 \right] \end{aligned} \tag{24}$$

From (24), the bias of $\bar{y}_{(P)}$, is given by

$$B(\bar{y}_{(P)}) \approx (S_1 - 1)\bar{Y} + S_1\bar{Y}\Upsilon \left[\frac{15}{8} g^2 C_x^2 - \frac{3}{2} g C_{yx} \right] + S_2 \frac{1}{2} \bar{X} g\Upsilon C_x^2 + S_3 \frac{1}{2} \bar{R} g\Upsilon C_{xr}. \tag{25}$$

The estimated bias of $\bar{y}_{(P)}$, is given by

$$\hat{B}(\bar{y}_{(P)}) \approx (S_1 - 1)\bar{y} + S_1\bar{y}\Upsilon \left[\frac{15}{8} g^2 C_x^2 - \frac{3}{2} g \frac{s_{yx}}{\bar{y}\bar{X}} \right] + S_2 \frac{1}{2} \bar{X} g\Upsilon C_x^2 + S_3 \frac{1}{2} \bar{R} g\Psi C_{xr}. \tag{26}$$

Subtracting estimated bias given in (26) from the proposed estimator given in (23), the unbiased proposed estimator becomes:

$$\bar{y}_{(P)}^{(U)} = \bar{y}_{(P)} - \hat{B}(\bar{y}_{(P)}) \tag{27}$$

or

$$\begin{aligned} \bar{y}_{(P)}^{(U)} = \quad & \left[S_1\bar{y} \left(\frac{c\bar{X} + d}{c\bar{x} + d} \right) + S_2(\bar{X} - \bar{x}) + S_3(\bar{R} - \bar{r}) \right] \exp \left(\frac{c(\bar{X} - \bar{x})}{c(\bar{X} + \bar{x}) + 2d} \right) \\ & - (S_1 - 1)\bar{y} - S_1\bar{y}\Upsilon \left[\frac{15}{8} g^2 C_x^2 - \frac{3}{2} g \frac{s_{yx}}{\bar{y}\bar{X}} \right] - S_2 \frac{1}{2} \bar{X} g\Upsilon C_x^2 - S_3 \frac{1}{2} \bar{R} g\Upsilon C_{xr} \end{aligned} \tag{28}$$

Solving (28) in terms of errors, we have

$$\begin{aligned} \bar{y}_{(P)}^{(U)} - \bar{Y} = \quad & \bar{Y}\Psi_0 + S_1\bar{Y} \left[-\frac{15}{8} \Upsilon g^2 C_x^2 \Psi_0 - \frac{3}{2} g\Psi_1 + \frac{3}{2} C_{yx}\Upsilon g\Psi_3 + \frac{3}{2} \Upsilon g C_{yx} \right. \\ & \left. - \frac{15}{8} \Upsilon g^2 C_x^2 + \frac{15}{8} g^2\Psi_1^2 - \frac{3}{2} g\Psi_0\Psi_1 \right] - S_2\bar{X} \left[\Psi_1 - \frac{1}{2} g\Psi_1^2 + \frac{1}{2} \Upsilon g C_x^2 \right] \\ & - S_3\bar{R} \left[\Psi_2 - \frac{1}{2} g\Psi_1\Psi_2 + \frac{1}{2} g\Upsilon C_{xr} \right] \end{aligned} \tag{29}$$

From (29), we have

$$B(\bar{y}_{(P)}^{(U)}) = 0. \tag{30}$$

Squaring (29) and then taking expectation, we get the variance of $\bar{y}_{(P)}^{(U)}$ as:

$$
\begin{aligned}
Var(\bar{y}_{(P)}^{(U)}) \approx \quad & \Upsilon \bar{Y}^2 C_y^2 + S_1^2 \bar{Y}^2 q_1 + S_2^2 \bar{X}^2 q_2 + S_3^2 \bar{R}^2 q_3 + 2S_1 \bar{Y}^2 q_4 - 2S_2 \bar{Y} \bar{X} q_5 \\
& -2S_3 \bar{Y} \bar{Z} q_6 - 2S_1 S_2 \bar{Y} \bar{X} q_7 - 2S_1 S_3 \bar{Y} \bar{R} q_8 + 2S_2 S_3 \bar{R} \bar{X} q_9,
\end{aligned}
\tag{31}
$$

where

$$
\begin{aligned}
q_1 = \quad & \frac{225}{64} g^4 \Upsilon^3 C_x^4 C_y^2 + \frac{9}{4} \Upsilon g^2 C_x^2 + \frac{9}{4} \Upsilon^3 g^2 C_{yx}^2 \left(\frac{\Delta_{220}}{\rho_{yx}} - 1 \right) - \frac{9}{4} \Upsilon^2 g^2 C_{yx}^2 - \frac{225}{64} \Upsilon^2 g^4 C_x^4 \\
& -\frac{45}{8} \Upsilon^3 g^3 C_x^2 C_{yx} C_y \frac{\Delta_{210}}{\rho_{yx}} - \frac{9}{2} \Upsilon^2 g^2 C_{yx} C_x \frac{\Delta_{120}}{\rho_{yx}} + \frac{45}{4} \Upsilon^2 g^3 C_{yx} C_x^2,
\end{aligned}
$$

$$
q_2 = \Upsilon C_x^2 \left(1 - \frac{1}{2} \Upsilon g^2 C_x^2 \right),
$$

$$
q_3 = \Upsilon \left(C_r^2 - \frac{1}{2} \Upsilon g^2 C_{xr}^2 \right),
$$

$$
q_4 = -\frac{15}{8} \Upsilon^2 g^2 C_y^2 C_x^2 - \frac{3}{2} \Upsilon g C_{yx} + \frac{3}{2} \Upsilon^2 g C_{yx} C_y \frac{\Delta_{210}}{\rho_{yx}},
$$

$$
q_5 = \Upsilon C_{yx},
$$

$$
q_6 = \Upsilon C_{yr},
$$

$$
q_7 = -\frac{15}{8} \Upsilon^2 g^2 C_{yx} C_x^2 - \frac{3}{2} \Upsilon g C_x^2 + \frac{3}{2} \Upsilon^2 g C_{yx} C_x \frac{\Delta_{120}}{\rho_{yx}} + \frac{1}{2} \Upsilon^2 g^3 C_x^4,
$$

$$
q_8 = -\frac{15}{8} \Upsilon^2 g^2 C_x^2 C_{yr} - \frac{3}{2} \Upsilon g C_{xr} + \frac{3}{2} \Upsilon^2 g C_{yx} C_r \frac{\Delta_{102}}{\rho_{yr}} + \frac{15}{16} \Upsilon^2 g^3 C_x^2 C_{xr} - \frac{3}{4} \Upsilon^2 g^2 C_{yx} C_{xr},
$$

$$
q_9 = \Upsilon C_{xr} - \frac{1}{4} \Upsilon^2 g^2 C_x^2 C_{xr}.
$$

The optimum values of $S_i (i = 1,2,3)$ are $S_{1(opt)} = -\frac{\mho_2}{\mho_1}$, $S_{2(opt)} = \frac{\bar{Y}}{\bar{X}} \left(\frac{\mho_3}{\mho_1} \right)$, $S_{3(opt)} = \frac{\bar{Y}}{\bar{R}} \left(\frac{\mho_4}{\mho_1} \right)$, where

$$
\mho_1 = q_1 q_2 q_3 - q_1 q_9^2 - q_2 q_8^2 - q_3 q_7^2 + 2q_7 q_8 q_9,
$$

$$
\mho_2 = q_2 q_3 q_4 - q_2 q_6 q_8 - q_3 q_5 q_7 - q_4 q_9^2 + q_5 q_8 q_9 + q_6 q_7 q_9,
$$

$$
\mho_3 = q_1 q_3 q_5 - q_1 q_6 q_9 - q_3 q_4 q_7 + q_4 q_8 q_9 - q_5 q_8^2 + q_6 q_7 q_8,
$$

$$
\mho_4 = q_1 q_2 q_6 - q_1 q_5 q_9 - q_2 q_4 q_8 + q_4 q_7 q_9 + q_5 q_7 q_8 - q_6 q_7^2.
$$

Substituting the optimum values of $S_i(i = 1,2,3)$ in (31), we get the minimum variance of $\bar{y}_{(P)}^{(U)}$, which is given by

$$Var(\bar{y}_{(P)}^{(U)})_{\min} \approx \bar{Y}^2\left[\Upsilon C_y^2 - \frac{\mho_5}{\mho_1}\right], \tag{32}$$

where

$$\mho_5 = \mho_{5a} + \mho_{5b},$$

$$\mho_{5a} = q_1 q_2 q_6^2 + q_2 q_3 q_4^2 - 2q_2 q_5 q_6 q_8 + q_1 q_3 q_5^2 - 2q_1 q_5 q_6 q_9 - 2q_3 q_4 q_5 q_7,$$

$$\mho_{5b} = -q_4^2 q_9^2 + 2q_4 q_5 q_8 q_9 + 2q_4 q_6 q_7 q_9 - q_5^2 q_8^2 + 2q_5 q_6 q_7 q_8 - q_6^2 q_7^2.$$

5. Numerical example

We use the following 4 real data sets for a numerical study.

Population 1: [source: [13]]

Y = Number of tube wells, X = Net irrigated area.

$N = 69$, $n = 10$, $\bar{Y} = 135.2609$, $\bar{X} = 345.7536$, $\bar{R} = 34.9565$, $\bar{K}^{(0)} = 0.4246$, $\bar{K}^{(1)} = 0.4981$, $\bar{K}^{(2)} = 0.4804$, $C_y = 0.8421$, $C_x = 0.8478$, $C_r = 0.5731$, $\rho_{yx} = 0.9224$, $\rho_{yr} = 0.7140$, $\rho_{xr} = 0.8193$, $\Delta_{220} = 8.0922$, $\Delta_{210} = 2.1398$, $\Delta_{120} = 2.1183$, $\Delta_{102} = 0.3920$, $\beta_{1x} = 2.3808$, $\beta_{2x} = 7.2159$.

Table 1. Results of different estimators for Population 1.

Estimator	PRE	Parameters		PRE values of three estimators		
		c	d	$\bar{y}_{(CK)}^{(U)}$	$\bar{y}_{(I)}^{(U)}$	$\bar{y}_{(P)}^{(U)}$
$\bar{y}_{(0)}^{(u)}$	100.00	1	C_x	335.54	738.08	858.78
$\bar{y}_{(HR)}^{(U)}$	561.22	1	β_{1x}	334.81	737.35	856.32
$\bar{y}_{(S_1)}^{(U)}$	359.21	1	β_{2x}	332.55	735.15	848.92
$\bar{y}_{(S_2)}^{(U)}$	403.69	1	ρ_{yx}	335.51	738.04	858.66
$\bar{y}_{(D)}^{(U)}$	695.01	C_x	1	335.39	737.92	858.24
		C_x	β_{1x}	334.61	737.15	855.64
		C_x	β_{2x}	331.95	734.58	847.04
		C_x	ρ_{yx}	335.43	737.96	858.39
		β_{1x}	1	335.75	738.28	859.47
		β_{1x}	C_x	335.78	738.31	859.58
		β_{1x}	β_{2x}	334.50	737.05	855.29
		β_{1x}	ρ_{yx}	335.77	738.29	859.53
		β_{2x}	1	335.90	738.41	859.93
		β_{2x}	C_x	335.90	738.42	859.97
		β_{2x}	β_{1x}	335.79	738.32	859.62
		β_{2x}	ρ_{yx}	335.89	738.42	859.95
		ρ_{yx}	1	335.43	737.96	858.39
		ρ_{yx}	C_x	335.51	738.04	858.66
		ρ_{yx}	β_{1x}	334.72	737.26	856.00
		ρ_{yx}	β_{2x}	332.27	734.88	848.03

https://doi.org/10.1371/journal.pone.0270277.t001

Population 2: [source: [13]]

Y = Number of tube wells, X = Number of tractors.

$N = 69$, $n = 10$, $\bar{Y} = 135.2609$, $\bar{X} = 21.2319$, $\bar{R} = 34.1159$, $\bar{K}^{(0)} = 7.1821$, $\bar{K}^{(1)} = 8.0432$, $\bar{K}^{(2)} = 6.3598$, $C_y = 0.8421$, $C_x = 0.7969$, $C_r = 0.8654$, $\rho_{yx} = 0.9118$, $\rho_{yr} = 0.7409$, $\rho_{xr} = 0.8654$, $\Delta_{220} = 6.7066$, $\Delta_{210} = 1.9555$, $\Delta_{120} = 1.8119$, $\Delta_{102} = 0.3825$, $\beta_{1x} = 1.855$, $\beta_{2x} = 3.7632$.

Population 3: [source: [14]]

This data set is based on Marmara region of Turkey in 2007.

Y = Number of teachers, X = Number of classes.

$N = 127$, $n = 31$, $\bar{Y} = 703.74$, $\bar{X} = 498.28$, $\bar{R} = 63.8897$, $\bar{K}^{(0)} = 1.2071$, $\bar{K}^{(1)} = 0.8906$, $\bar{K}^{(2)} = 1.0663$, $C_y = 1.2559$, $C_x = 1.1150$, $C_r = 0.5769$, $\rho_{yx} = 0.9789$, $\rho_{yr} = 0.8312$, $\rho_{xr} = 0.8516$, $\Delta_{220} = 4.7079$, $\Delta_{210} = 1.6136$, $\Delta_{120} = 1.6171$, $\Delta_{102} = 0.4233$, $\beta_{1x} = 1.7205$, $\beta_{2x} = 2.3149$.

Population 4: [source: [14]]

This data set is based on Marmara region of Turkey in 2007.

Y = Number of teachers, X = Number of students.

$N = 127$, $n = 31$, $\bar{Y} = 703.74$, $\bar{X} = 20804.59$, $\bar{R} = 64.0$, $\bar{K}^{(0)} = 0.0433$, $\bar{K}^{(1)} = 0.0296$, $\bar{K}^{(2)} = 0.0295$, $C_y = 1.2559$, $C_x = 1.4654$, $C_r = 0.5751$, $\rho_{yx} = 0.9366$, $\rho_{yr} = 0.8240$, $\rho_{xr} = 0.7834$, $\Delta_{220} = 4.7079$, $\Delta_{210} = 1.5674$, $\Delta_{120} = 1.7115$, $\Delta_{102} = 0.4015$, $\beta_{1x} = 2.1638$, $\beta_{2x} = 4.5928$.

The results based on Populations 1–4 are given in Tables 1–4. Tables 1–4 give the results when no conventional measures and conventional measures are used. We use the following

Table 2. Results of different estimators for Population 2.

Estimator	PRE	Parameters		PRE values of three estimators		
		c	d	$\bar{y}_{(CK)}^{(U)}$	$\bar{y}_{(I)}^{(U)}$	$\bar{y}_{(P)}^{(U)}$
$\bar{y}_{(0)}^{(u)}$	100.00	1	C_x	309.18	641.20	752.07
$\bar{y}_{(HR)}^{(U)}$	519.04	1	β_{1x}	303.34	636.46	738.04
$\bar{y}_{(S_1)}^{(U)}$	402.10	1	β_{2x}	293.93	629.77	718.67
$\bar{y}_{(S_2)}^{(U)}$	589.26	1	ρ_{yx}	308.52	640.64	750.39
$\bar{y}_{(D)}^{(U)}$	627.44	C_x	1	306.59	639.04	745.63
		C_x	β_{1x}	300.89	634.60	732.62
		C_x	β_{2x}	289.65	627.10	711.00
		C_x	ρ_{yx}	307.20	639.55	747.13
		β_{1x}	1	310.68	642.50	755.94
		β_{1x}	C_x	311.33	643.07	757.65
		β_{1x}	β_{2x}	302.43	635.76	735.99
		β_{1x}	ρ_{yx}	310.96	642.75	756.68
		β_{2x}	1	312.31	643.95	760.29
		β_{2x}	C_x	312.64	644.24	761.18
		β_{2x}	β_{1x}	310.95	642.74	756.66
		β_{2x}	ρ_{yx}	312.45	644.07	760.68
		ρ_{yx}	1	307.47	639.77	747.79
		ρ_{yx}	C_x	308.73	640.83	750.94
		ρ_{yx}	β_{1x}	302.40	635.74	735.93
		ρ_{yx}	β_{2x}	292.28	628.71	715.62

Table 3. Results of different estimators for Population 3.

Estimator	PRE	Parameters		PRE values of three estimators		
		c	d	$\bar{y}_{(CK)}^{(U)}$	$\bar{y}_{(I)c}^{(U)}$	$\bar{y}_{(P)}^{(U)}$
$\bar{y}_{(0)}^{(u)}$	100.00	1	C_x	1068.26	2449.65	4339.58
$\bar{y}_{(HR)}^{(U)}$	1108.83	1	β_{1x}	1067.57	2449.38	4328.87
$\bar{y}_{(S_1)}^{(U)}$	460.04	1	β_{2x}	1066.90	2449.12	4318.48
$\bar{y}_{(S_2)}^{(U)}$	729.92	1	ρ_{yx}	1068.42	2449.72	4342.01
$\bar{y}_{(D)}^{(U)}$	2397.13	C_x	1	1068.51	2449.75	4343.48
		C_x	β_{1x}	1067.77	2449.46	4332.00
		C_x	β_{2x}	1067.17	2449.22	4322.64
		C_x	ρ_{yx}	1068.53	2449.76	4343.82
		β_{1x}	1	1068.87	2449.89	4349.14
		β_{1x}	C_x	1068.79	2449.87	4347.94
		β_{1x}	β_{2x}	1068.00	2449.55	4335.49
		β_{1x}	ρ_{yx}	1068.88	2449.90	4349.37
		β_{2x}	1	1069.04	2449.96	4351.84
		β_{2x}	C_x	1068.98	2449.94	4350.94
		β_{2x}	β_{1x}	1068.68	2449.82	4346.23
		β_{2x}	ρ_{yx}	1069.05	2449.97	4352.08
		ρ_{yx}	1	1068.37	2449.70	4341.23
		ρ_{yx}	C_x	1068.23	2449.64	4339.16
		ρ_{yx}	β_{1x}	1067.53	2449.37	4328.22
		ρ_{yx}	β_{2x}	1066.84	2449.10	4317.62

expression to obtain the percent relative efficiency (PRE) as:

$$\text{PRE} = \frac{Var(\bar{y}_{(0)}^{(U)})}{Var(\bar{y}_{(i)}^{(U)})} \times 100,$$

where $i = 0, HR, S_1, S_2, D, CK, I, P$.

In Tables 1–4, the proposed unbiased estimator $\bar{y}_{(P)}^{(U)}$ outperforms in all four Populations but the [4] estimators $\bar{y}_{(S_1)}^{(U)}$ in Populations 1–3 and [1] estimator $\bar{y}_{(HR)}^{(U)}$ in Population 4 are performing poorly.

6. Conclusion

In this study, we have proposed an unbiased class of estimators in estimating the finite population mean in simple random sampling using the single auxiliary variable and rank of the auxiliary variable. Expressions for biases and MSEs or variances are obtained up to first order of approximation. Four data sets are used for numerical study. The proposed estimator outperforms in all four populations as compared to all considered estimators. It is observed that use of conventional measures i.e. $C_x, \beta_{1x}, \beta_{2x},$ and ρ_{yx} have significant role in increasing the efficiency of the estimators in Tables 1–4.

[4] estimators $\bar{y}_{(S_1)}^{(U)}$ in Populations 1–3 and [1] estimator $\bar{y}_{(HR)}^{(U)}$ in Population 4 show the poor performance but the proposed unbiased estimator $\bar{y}_{(P)}^{(U)}$ have an excellent performance as compared to all considered estimators in all four populations 1–4.

Table 4. Results of different estimators for Population 4.

Estimator	PRE	Parameters		PRE values of three estimators		
		c	d	$\bar{y}_{(CK)}^{(U)}$	$\bar{y}_{(I)}^{(U)}$	$\bar{y}_{(P)}^{(U)}$
$\bar{y}_{(0)}^{(u)}$	100.00	1	C_x	688.84	964.64	1040.86
$\bar{y}_{(HR)}^{(U)}$	230.94	1	β_{1x}	688.83	964.64	1040.78
$\bar{y}_{(S_1)}^{(U)}$	769.77	1	β_{2x}	688.80	964.64	1040.49
$\bar{y}_{(S_2)}^{(U)}$	773.16	1	ρ_{yx}	688.85	964.64	1040.92
$\bar{y}_{(D)}^{(U)}$	983.47	C_x	1	688.85	964.64	1040.95
		C_x	β_{1x}	688.84	964.64	1040.86
		C_x	β_{2x}	688.82	964.64	1040.66
		C_x	ρ_{yx}	688.85	964.64	1040.96
		β_{1x}	1	688.86	964.64	1040.78
		β_{1x}	C_x	688.83	964.64	1040.98
		β_{1x}	β_{2x}	688.86	964.64	1040.78
		β_{1x}	ρ_{yx}	688.86	964.64	1041.01
		β_{2x}	1	688.86	964.64	1040.99
		β_{2x}	C_x	688.86	964.64	1040.98
		β_{2x}	β_{1x}	688.86	964.64	1040.98
		β_{2x}	ρ_{yx}	688.86	964.64	1041.01
		ρ_{yx}	1	688.85	964.64	1040.91
		ρ_{yx}	C_x	688.84	964.64	1040.85
		ρ_{yx}	β_{1x}	688.83	964.64	1040.76
		ρ_{yx}	β_{2x}	688.79	964.63	1040.46

https://doi.org/10.1371/journal.pone.0270277.t004

Acknowledgments

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