Some properties of the dynamics \hspace{1em}
In all the following, we use the notation $x \in L^\infty(\Omega \times \mathcal{C}_{opp})$ instead of letter $a$ to refer to the neural activity, because it is mathematically more rigorous to speak of a functional equation on the Banach space $L^\infty(\Omega \times \mathcal{C}_{opp})$. The neural activity $x$ is solution to

$$\frac{dx}{dt} = -x(t) + F(\omega \ast x(t) + H) =: \Theta(x(t)),$$

with $a(r, c, t) := x(t)(r, c)$. Recall that $H$ is also supposed to be constant w.r.t. time.

**Lemma 1** (Condition for the existence of a unique stationary solution). Let $d \in \{1, 2, 3\}$ denote the color space dimension. Suppose that

$$F'(0) \int_{\mathbb{R}^2} |g| \int_{\mathbb{R}^3} (|f_1| + |f_2|) \|x\| \leq 1. \tag{1}$$

Then there exists a unique stationary solution to Eq 3 (main text) in $L^\infty$.

More precisely, for $H \in L^\infty(\Omega \times \mathcal{C}_{opp})$, the map

$$\Phi_H : \left( L^\infty(\Omega \times \mathcal{C}_{opp}) \xrightarrow{x} L^\infty(\Omega \times \mathcal{C}_{opp}) \right) \mapsto F(\omega \ast x + H),$$

is Lipschitz continuous, with the Lipschitz constant given in the left hand side of Eq (1), ensuring $\Phi_H$ to be a contraction with respect to $L^\infty$ norm.

**Proof.** For any $x, y \in L^\infty(\Omega \times \mathcal{C}_{opp}),$

$$\|\omega \ast x\|_\infty \leq \int_{\mathbb{R}^2} |g| \int_{\mathbb{R}^3} (|f_1| + |f_2|) \|x\|_\infty.$$

Indeed, for any $(r, c) \in \Omega \times \mathcal{C}_{opp},$

$$|\omega \ast x|(r, c) \leq \int_{\Omega} |g(r - r')| dr' \int_{\mathcal{C}_{opp}} |f_1(c - c') - f_2(c + c')| dc' \|x\|_\infty \leq \int_{\mathbb{R}^2} |g| \int_{\mathbb{R}^3} (|f_1| + |f_2|) \|x\|_\infty.$$

Thus, for $x, y,$

$$\|\Phi_H x - \Phi_H y\|_\infty \leq F'(0) \int_{\mathbb{R}^2} |g| \int_{\mathbb{R}^3} (|f_1| + |f_2|) \|x - y\|_\infty. \tag{2}$$

$\Box$
In fact, the same conditions ensure linear stability of the solution, which is the object of the next lemma. Let \( \mathcal{E} \) denote \( L^\infty(\Omega \times \mathcal{C}_{\text{opp}}) \).

**Lemma 2 (Stability).** Under the conditions of Lemma 1, the unique stationary solution is linearly stable.

**Proof.** Let \( x_0 \) denote the stationary solution. The linearization of \( \Theta \) around it gives

\[
D\Theta(x_0) \cdot x = -x + F'(\omega \star x_0 + H) \omega \star x \in \mathcal{L}(\mathcal{E}, \mathcal{E}).
\]

Let \( \mathcal{L} := D\Theta(x_0) \) denote the linear part. Then, \( \mathcal{L} = -Id + \mathcal{T} \) where

\[
\mathcal{T} := F'(\omega \star x_0 + H) \omega \star
\]

is a linear operator such that \( \|\mathcal{T}\| < 1 \) thanks to condition (1). Note that \( \mathcal{T} \) takes values in \( \mathcal{C}_0(\Omega \times \mathcal{C}_{\text{opp}}) \) the set of continuous functions defined on the domain. The spectrum of \( \mathcal{L} \), denoted \( \Sigma(\mathcal{L}) := \{ \sigma \in \mathbb{C} \mid \mathcal{L} - \sigma Id \text{ not bijective} \} \), is then equal to \(-1 + \Sigma(\mathcal{T})\), which is a compact contained in a disk centered on \(-1\) and of radius \( \|\mathcal{T}\| \). Thus, for any \( \sigma \in \Sigma(\mathcal{L}) \) we get that \( \Re \sigma < 0 \), which ensures linear stability.

Notice that this does not imply global convergence of the dynamics to the unique stationary solution.

**Lemma 3.** Let \( d \) denote the dimension of the color space, \( g_1 \) and \( g_2 \) the two gaussians such that \( g = g_1 - g_2 \) and \( \mathcal{D} \) the closed disk on which \( g_1 \geq g_2 \). The radius of the disk is given by

\[
r_0 := \sqrt{\frac{2}{1/\alpha^2 - 1/\beta^2} \log \frac{\mu}{\nu}}.
\]

The contraction condition (1) is equivalent to

\[
\frac{3}{4} \left[ \int_{\mathcal{D}} (g_1 - g_2) - \int_{\mathbb{R}^2 \backslash \mathcal{D}} (g_1 - g_2) \right] \int_{\mathbb{R}^d} (f_1 + f_2) < 1
\]

where

\[
\int_{\mathbb{R}^d} u = \mu_c (2\pi)^{d/2} \alpha_c^d
\]

and where the bracket is equal to

\[
2\pi \mu \alpha^2 \left( 1 - 2 \left( \frac{\mu}{\nu} \right)^{-\frac{3}{1-\alpha/\beta^2}} \right) - 2\pi \nu \beta^2 \left( 1 - 2 \left( \frac{\mu}{\nu} \right)^{-\frac{1}{\alpha/\beta^2}} \right)
\]

thanks to the formulas \( \int_{\mathcal{D}} g_1 = 2\pi \mu \alpha^2 (1 - e^{-\frac{r_0^2}{2\sigma_1^2}}) \) and \( \int_{\mathbb{R}^2} g_1 = 2\pi \mu \alpha^2 \).
Color matching as a projection

**Lemma 4.** Suppose that $J^{\text{comp}}[c]$ is smooth function of $c$, and that condition (1) holds. Then the unique stationary solution $a[c]$ to the dynamics with input $H[c]$ related to $J^{\text{comp}}[c]$ is smoothly parameterized by $c$. Hence under these assumptions, **color matching consists in projecting** $a^{\text{test}}$ **on the image set of the parameterization** $\{a^{\text{comp}}[c]\}$.

**Proof.** For any $c \in \mathcal{C}$, the unique stationary solution $a[c]$ satisfies $0 = Q(a[c],c)$ where the map $Q$ is defined as

$$Q : \left( L^\infty \times \mathcal{C}, (a,c) \mapsto -a + F(\omega \ast a + H[c]) \right).$$

For $J^{\text{comp}}[\cdot]$ regular enough, $Q$ is $C^k$ on $L^\infty \times \mathcal{C}$, and the partial differential $D_a Q(a, c)$ defined below is invertible:

$$D_a Q(a, c) \cdot da = -da + F'(\omega \ast a + H[c]) \omega \ast da$$

because for any $b \in L^\infty$, $da \mapsto F'(\omega \ast a + H[c]) \omega \ast da - b$ defines a contraction mapping in $L^\infty$ under condition (1) (we used the fact that $|F'| \leq F'(0)$), and we can apply Picard’s theorem. Then, in a neighborhood of each $c_0$ and $a[c_0]$ the map $c \mapsto a[c]$ is $C^k$ thanks to the Implicit Function Theorem. We thus obtain a smoothly parameterized family of elements in $L^\infty(\mathcal{C}) \{a[c]\}_{c \in \mathcal{C}}$. 

$\square$