In this section, we will establish that

\[ SMR(\hat{T}_{fn}) \leq \beta + o(1). \]  

(S1)

Define \( \hat{j} = \min\{j \geq 1 : p(\hat{\pi}d + j) \leq F_{\hat{\pi},(j)}^{-1}(\beta)\} \cdot 1_{\{\hat{\pi}d > t_\alpha\}} \), then \( \hat{T}_{fn} = \hat{\pi}d + \hat{j} \). Also define \( n(j) = j - s(j) \) as the number of noncausal variants among the top \( j \) ranked variants. We have

\[
SMR(\hat{T}_{fn}) = P(s(\hat{T}_{fn}) < (1 - \epsilon)s) = P(n(\hat{T}_{fn}) > \hat{T}_{fn} - (1 - \epsilon)s) \\
= P(n(\hat{T}_{fn}) > \hat{\pi}d + \hat{j} - (1 - \epsilon)\pi d) \\
\leq P(n(\hat{T}_{fn}) > \hat{\pi}d + \hat{j} - (1 - \epsilon)\pi d, \hat{\pi} \geq (1 - \epsilon)\pi) + P(\hat{\pi} < (1 - \epsilon)\pi) \\
\leq P(n(\hat{T}_{fn}) > \hat{j}) + o(1), \tag{S2}
\]

where the second equality is by \( \hat{T}_{fn} = n(\hat{T}_{fn}) + s(\hat{T}_{fn}) \), the third equality is by \( s = \pi d \), and the last step is by the consistency of \( \hat{\pi} \).

When \( \hat{\pi}d \leq t_\alpha \), we have \( \hat{j} = 0 \) and \( \hat{T}_{fn} = \hat{\pi}d \). Then

\[
P(n(\hat{T}_{fn}) > \hat{j}) = P(n(\hat{\pi}d) > 0) \leq P(n(t_\alpha) > 0) = \alpha \tag{S3}
\]

When \( \hat{\pi}d > t_\alpha \), denote \( P^0_{(j)} \) as the \( j \)th ordered \( p \)-value of \( n - s \) noncausal variants, then

\[
P(n(\hat{T}_{fn}) > \hat{j}) = P(P^0_{(j)} < p_{(\hat{T}_{fn})}) \leq P(P^0_{(j)} < F_{\hat{\pi},(j)}^{-1}(\beta)) \leq P(P^0_{(j)} < F_{\hat{\pi},(j)}^{-1}(\beta), \hat{\pi} \leq \pi) + P(\hat{\pi} > \pi) \leq P(P^0_{(j)} < F_{\pi,(j)}^{-1}(\beta)) + o(1). \]

By the definition of \( F^{-1} \), (S1) follows.