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A typology of rules for knowledge exchange in higher-order interactions

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Abstract

Social learning is important to humans and other animals as they gather information about their environment. Information and behaviours can therefore spread rapidly through social networks as contagions. However, the way individuals acquire and use social information is highly variable and frequently complex, often shaped by higher-order or multibody interactions that are not straightforwardly described by conventional dyadic networks. There has been considerable recent progress in modeling social contagions across higher-order networks that explicitly quantify these multibody interactions. A challenge for studying social contagion across multi-body or higher-order interactions is the diversity of ways in which knowledge can be exchanged within or among groups. Here we provide a typology of knowledge exchange rules in higher-order networks, focusing on both learning and discovery. We also provide a non-exhaustive list of many basic knowledge exchange rules, to demonstrate our typology and its value in distinguishing between different mechanisms of social learning. Our aim is to provide a framework that helps researchers interested in modeling knowledge exchange in higher-order networks to develop new models and adapt existing models to questions of interest. By doing so we hope to promote interdisciplinarity in the study of how multibody interactions shape social contagions, especially at this critically incipient stage - avoiding the inevitable challenges from the eventual need to integrate parallel, independent, complementary advances among different disciplines.

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Author summary

Learning from each other is important to humans and other animals as it provides safe or quick ways to gather information about the world around you. Because of this ‘social learning’, ideas and behaviours can spread quickly through our social networks. Previously, researchers have mainly described these social networks using ‘dyadic’ approaches that only consider connections between pairs of individuals at a time. However, often social interactions occur between more than two individuals at a time, and this can influence how we learn from each other. Here we classify different ways that social learning occurs in groups and provide a list of and mathematical formulae for many potential rules for how knowledge is exchanged in these contexts. We aim to provide a framework for researchers interested in modeling how we learn from each other in groups, and help to integrate ideas from different areas of research into social learning. Doing so now is valuable as the number of studies using higher-order networks to study group behviour is growing fast.

Introduction

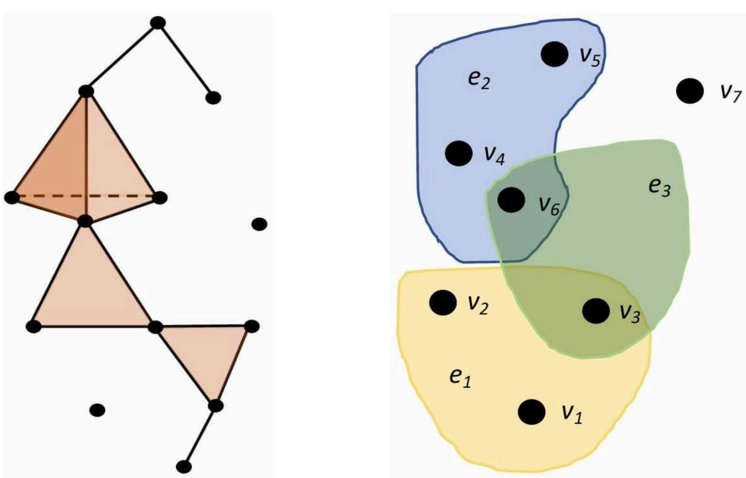
Individuals in any society do not learn about the world around them in isolation. Social learning is commonly used by humans and other animals to gather information faster and without some of the same risks as individual learning [1,12,13,23]. Consequently, we have learned a great deal about how knowledge is exchanged within populations by quantifying who interacts with whom [1,5,31]. Much of the recent progress has used tools from social network analysis and modelling that allow us to capture the properties of societies that facilitate the spread of useful information [9,11,31].

The rules that govern how individuals acquire and use information they gather from those they communicate with are complex [20,21]. Individuals may favour knowledge provided by trusted or successful partners [15]. They may also be more likely to use information suggested by a greater proportion of their connections or to copy the majority resulting in conformist learning that follows a threshold model [7,16,17]. As a result, the spread of information through networks is often considered a complex contagion, distinct from simple contagions in which there is an independent probability of learning with each interaction [4,11]. However, there are aspects of social learning that are difficult or impossible to capture using a network approach. This is especially true when individuals learn in groups of varying sizes that might influence the knowledge they gain or discoveries they make [2].

For complex contagions, it is important to consider higher-order (or multibody) interactions in groups, as the exchange of knowledge will depend on the simultaneous state of the multiple individuals interacting together [2,10]. We can imagine a situation, for example, where someone believes (erroneously) that you add cream and then jam when making a cream tea. While there may be a very small chance of them realising the error of their ways if just one friend tells them how wrong they are, the chances of them changing their behaviour for the better will be higher if a majority of their friends tell them they should be adding the jam first, and much higher still if all of their friends tell them at once in a chorus of disapproval. This provides a simple example of why considering higher-order interactions might be important for providing insights into understanding knowledge exchange beyond those provided by a network approach (in which the last two examples would be equivalent).

There are two widely used ways to mathematically represent higher-order interactions: simplicial sets [14,28] and hypergraphs [3,30]. Fig 1 provides a visual representation for each. Briefly, a simplicial set consists of a series of k-order simplices. A 0-simplex is equivalent to a vertex, or node, in a network and 1-simplex is the same as two vertices being connected by an edge. However, higher-order simplices allow larger groups to be considered, for example a 2-simplex would represent the interactions among three individuals (*A*,*B* and *C*) and 9-simplex would represent a group of 10 individuals ($A \rightarrow J$). Simplicial complexes (widely used in topology; [26]) are simplicial sets with an additional assumption of downward closure; that is that a simplicial complex containing a higher-order simplex will also necessarily contain the lower order simplices nested within it (e.g. if the 2-simplex *ABC* were to exist in a simplicial complex it would mean that the 1-simplices *AB*, *AC* and *BC* also existed) [28]. Hypergraphs provide an alternative route to quantifying higher-order interactions by connecting (any number of) individuals with a hyperedge [30]. In this paper, we choose to focus on notation from simplicial sets rather than a hypergraph approach as it provides us with a more flexible syntatic framework in which to define the rules of knowledge exchange (see [14]). In most real-world systems the two approaches will be practically equivalent, and approaches to modelling knowledge exchange using either approach are conceptually identical. Note that we write simplicial sets as X_j^k, where *k* is the order of the simplex and the *j* represents the unique ordering of that simplex within the simplicial set (note: this can be arbitrary so long as it is consistent). A more detailed introduction to simplicial sets, hypergraphs and simplicial complexes is provided by [3,28,30].

Because they capture different orders of interactions, the topology of simplicial sets can provide a wealth of information about the structure of a complex system, much of which is not apparent from dyadic network approaches alone [14].



a) Example of a simplicial set that has 0-simplices, 1-simplices, and 2-simplices

b) Example of a hypergraph, where the blue, green, and yellow shapes are the hyperedges (labeled e_1, e_2, e_3) and $v_{1,\ldots,7}$ are the vertices

Fig 1. **Visual representations of the two most common ways to mathematically model higher-order interactions: simplicial sets and hypergraphs.** In Fig 1a the 0-simplices are the black circles, the 1-simplices are two black circles connected by a black line, and the 2-simplices are three black circles connected by black lines and an orange triangle. In Fig 1b the black circles represent nodes/vertices in the hypergraph and the shaded areas represent the hyperedges that connect them. Note that the two panels do not illustrate the same social structure.

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Consider the following example: three students, Anna, Nina, and Matt, are all working together on a group project. If we use a conventional dyadic network to model their collaboration, there would be no difference in the model if the three students meet up and all work together at the same time or if Anna and Nina meet, Nina and Matt meet, and Anna and Matt meet separately. While there is no difference in the network, meeting in one large group can lead to very different ideas and knowledge exchange, than meeting pairwise. An additional dynamic we may want to consider is if there are distinct interactions happening within a group: what if the students met all together, and while they are working, Nina and Anna are also texting each other ideas? This could lead to a different group dynamic, but a dyadic network model alone would be unable to differentiate these cases. Thus, using a simiplical set is much more beneficial as we can explicitly model these different dynamics (see Fig 2). In general, simplicial sets allow for a more nuanced exploration of multibody communication, explicit sub-structure dynamics, and the effects of group characteristics on interactions (e.g. group size). Thus, moving from using dyadic networks to simplicial approaches that explicitly consider higher-order interactions provides an opportunity for new insights on a wide range of questions [10,14,18,19,22,25].

The added complexity of simplicial sets and hypergraphs means they allow for a richer set of potential knowledge exchange rules than dyadic networks. However, as the use of these approaches is still in its infancy, we lack a framework in which to consider these potential rules. Our aim here is to set out an initial typology of knowledge exchange rules for multibody interactions to promote interdiscplinary thinking and collaboration around modelling social learning in higher-order networks. We generate two broad types of knowledge exchange rule: learning and discovery, and then provide a system to further classify rules within these broad categories. To illustrate how our typology works, and promote discussion around modelling choices for knowledge exchange in higher-order networks, we then provide the mathematical formulation for a non-exhaustive subset of potential learning and discovery rules. We focus on notation for simplicial sets as these represent a more flexible approach than hypergraphs, and allow for more convenient mathematical notation for more complex knowledge exchange rules. We then illustrate how these example knowledge exchange rules would apply to a simplified, hypothetical higher-order structure and discuss how applying our typology can guide modelling decisions in future theoretical and empirical research.

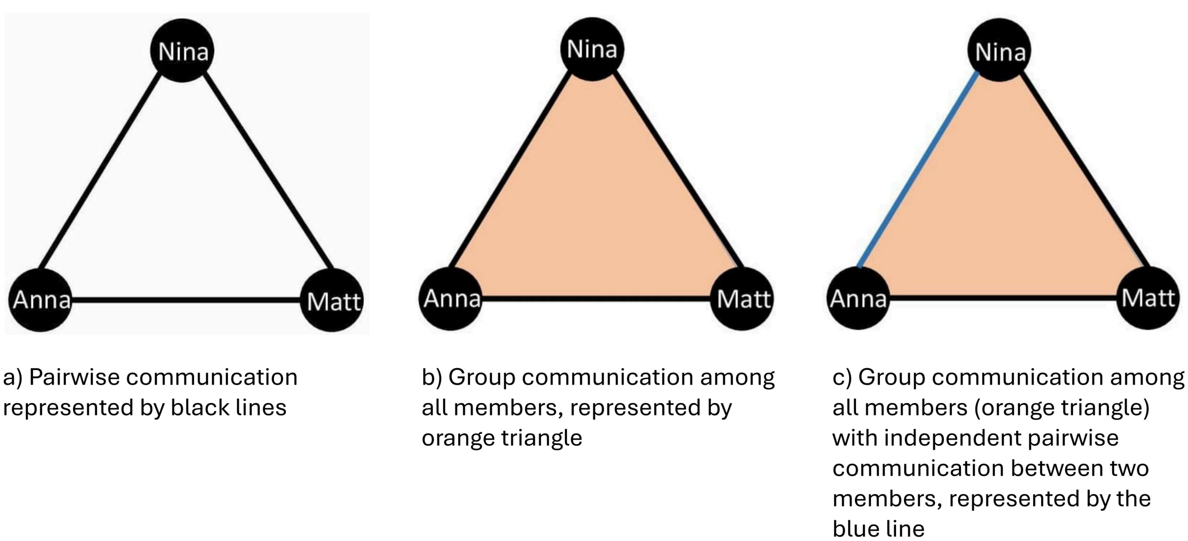

Fig 2. **The graphical representation of how simplicial sets can model the example of the three scenarios of students working on a group project: meeting pairwise, meeting all together, and meeting all together with independent pairwise communication between two members.** Note Fig 2a is how a network would represent all three scenarios.

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A classification of knowledge exchange rules by key features

We start by exploring some broad key features that can help define knowledge exchange rules (KER). This is by no means an exhaustive list and as this field continues to grow, and as more KER are developed, important features for these rules may be added or the ones provided here modified.

Feature 1: Learning versus discovery

Here we use learning and discovery to define two distinct ways in which individuals can exchange and use knowledge. We will start with ***learning***, which we define as a set of algorithms in which knowledge is gained (or lost) using "token-passing" algorithms (Fig 3a). These algorithms could be additive (an individual can gain or lose units of knowledge with each social interaction) or Boolean (an individual is either knowledgeable or not). As an example, learning would include a case in which a group contained a teacher and a set of students. Each student would gain knowledge from interacting in a group with a teacher.

Discovery, on the other hand, captures situations in which knowledge is gained in a non-additive or cumulative manner, i.e. the combination of two distinct pieces of information to gain something new (Fig 3b). For example, if a small group that knows how to grow hops interacts with another group who know fermentation, then only together they could they create beer. Note, that discovery need not always lead to progress. Consider a situation where two people with strongly polarised views on a topic were to engage with each other. At the end, both people might emerge with a different and more uncertain view. Whether the process being modelled follows the rules of learning or discovery is integral to shaping how it fits within our classification. Both provide their own set of (somewhat overlapping) decisions as to the most appropriate algorithm or model.

Feature 2: Categorical, ordinal or cardinal

Feature 2.1: What knowledge is being modeled? An important classification of the knowledge being exchanged by actors within a complex system is whether it is in categorical, ordinal or cardinal form. This property of the knowledge itself will influence the mathematical structure of the KER generated. Starting with **categorical**, which, at its simplest, reflects a binary state of knowledge in which an individual either possesses that knowledge or not (equivalent to classic compartmental models of contagions; [29]). Trivia can be an example of a type of categorical knowledge: you either know a fact or you don't. More complex forms of categorical knowledge can exist and are especially pertinent when considering rules related to discovery rather than learning. When using discovery rules different categories of knowledge can be combined to produce novel categories that did not previously exist. We will expand on this topic and introduce some examples of KER below.

We can also consider knowledge as an **ordinal** trait, with individuals able to move between knowledge "levels" based on their interactions within simplices. An example of this is children need to learn how to count before they can learn to add and they need to learn add before they can learn to multiply. So, gaining new knowledge (e.g., multiplication) depends on knowing previous knowledge (e.g., addition). While categorical and ordinal knowledge both treat knowledge as discrete categories, **cardinal knowledge** considers knowledge as a continuous trait. Learning rules would then allow individuals to gain (or lose) units of knowledge as defined by the size and membership of the simplices of which they are a member. Cardinal knowledge can be thought about as gaining expertise or experience in a topic. For a specific example, let's consider gaining expertise in literature. There is not a specific order in which books need to be read to become an expert and there isn't a discrete cutoff between expert and not. Note while both ordinal and cardinal knowledge are relevant when considering rules related to learning, there is no reason that sets of learning rules and discovery rules could not be combined together within a single model.

Feature 2.2: How is knowledge gained or lost?. We can also classify the change in knowledge for each individual as categorical, ordinal or cardinal. Categorical KER can apply naturally to sets of rules for both learning and discovery. As

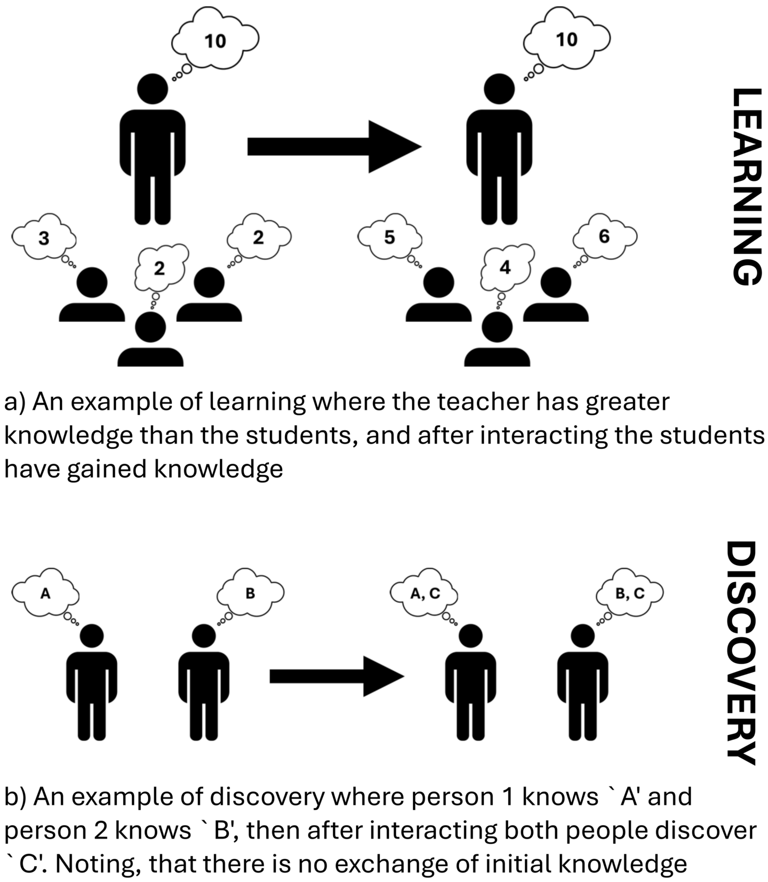

Fig 3. Visual examples of learning and discovery rules applied to hypothetical social interactions in humans. These multibody social interactions could be represented as either simplicial sets or hypergraphs.

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an example of the former, categories could be a simple as knowledge either increasing by a given amount or not changing, with category membership defined by the chosen rule. However, more complex combinations of categories each with their own associated rules could be applied. Ordinal and cardinal rules will allow greater flexibility in how individual knowledge can change. Ordinal rules allow multiple steps in how much individuals can gain (or lose) from particular interactions, perhaps associated with how many more knowledgeable individuals are contained within a set, for example. Cardinal rules would of course extend this to allow individuals to change their knowledge values by any value based on how their knowledge relates to that of other individuals within the set. An important note here is that while ordinal or categorical changes are possible for cardinal knowledge, and categorical change is possible for ordinal knowledge, the opposite combinations are not feasible (e.g. it is not possible for continuous change in knowledge when an individual is either knowledgeable or not).

Feature 3: Trajectory

A general feature of many knowledge exchange rules will be whether an individual can only gain knowledge or information over time, or whether they could also reduce their knowledge. These rules apply most clearly to learning rather than discovery, although can apply in a limited way to the latter also. When individuals can only gain knowledge from their interactions with others it results in all (or the vast majority of) individuals gaining knowledge over time and typically converging at high(er) levels of knowledge. Obvious examples of these type of learning rules being important would be in many educational or coaching scenarios where individuals are likely to gain knowledge from more knowledgeable individuals in their simplex through token-passing algorithms in which the more knowledgeable individuals retain the value of the tokens passed (see Fig 3a for an example). In contrast, sets of rules in which individuals can both gain and lose knowledge will typically result in convergence over time at an intermediate value. Rules such as these might be especially useful when trying to understand how individuals with opposing viewpoints form compromises when they interact within a series of groups. The potential rules within this set will naturally be more diverse than learning rules in which individuals can only gain (or only lose) knowledge over time.

Discovery rules may also contain different trajectories based on whether individuals retain or lose their existing knowledge. For example, we could assume that when individuals engage with each other they retain their existing knowledge and gain the new knowledge created through their interaction (see Fig 3b for an example). Or, instead, we could model a scenario where individuals do not necessarily retain their existing knowledge. In this scenario, their "discovery" could be a new viewpoint or greater uncertainty about the world. We classify these types of discovery as **interference rules**, i.e. the combination of two distinct pieces of information to gain something new, but lose the initial knowledge (see Fig 4). A simple example would be two people from opposing political parties meeting and discussing their politics, and leaving the conversation as independents unsure who to vote for.

Feature 4: Who learns when

An important aspect of any KER is determining when an individual changes their knowledge in a simplex. There are clearly a hugely diverse set of potential rules here, but we distinguish them along two major axes – who changes and when do they follow a particular trajectory. We group these two features together as they share a very similar set of considerations.

Feature 4.1: Who changes. Not all individuals need to learn at every opportunity (whether it be a time step or simplex). Each rule provided can be used to define the subset of individuals that learn (or discover) at specific time steps. Many rules are likely to apply to all individuals in a simplex, however, others may apply to specific individuals (e.g., the least knowledgeable or the most different) or smaller subsets of the group (e.g., those with below average knowledge).

Feature 4.2: What decides their trajectory. How the knowledge value of an individual relates to a subset of other individuals in a simplex can also define its trajectory of change (i.e., whether they increase or decrease in knowledge). For example, a simple learning rule could consist of all individuals but the smartest gaining one unit of knowledge, see Fig 3a as an example. However, these rules can also be more complex, e.g., an individual could gain one unit of knowledge for every individual in the group smarter than them. In both of the previous examples, the trajectory of all individuals in a simplex is calculated using the same function. However, if the trajectories of different subsets of individuals are calculated in different ways, then it is most appropriate to define these as distinct, interacting KERs. An example would be a situation

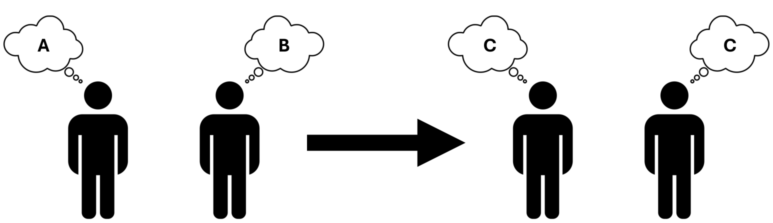

Fig 4. An example of an interference discovery rule where person 1 knows 'A' and person 2 knows 'B', then after interacting both people discover 'C', but no longer know (or are certain of) 'A' and 'B' respectively.

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where individuals with more knowledge than a fixed proportion of their group mates lose knowledge, and individuals with less knowledge than a/the threshold gain knowledge. This simplifies the specification of the rules defined and algorithms used, and provides greater flexibility as to when different parts of a rule set apply. For example, in the case above, while it possible to write the most simple version as a single learning rule, it offers greater flexibility to write it as two separate rules and potentially allow different thresholds or different magnitudes of knowledge change.

One exception to the previous statement is when changes in individual knowledge are inherently linked. For instance, token-passing algorithms could be seen as genuine exchanges, whereby one subset of individuals directly passes knowledge to a second subset of individuals. In an example of one such token-passing learning rule, the most knowledgeable individual in a set could lose x units of knowledge that are then redistributed among n individuals with below average knowledge so that each individual in this subset received x/n units of knowledge. Cases such as this are best represented as a single learning rule being applied. Further the trajectories themselves can be defined in numerous ways as captured by Feature 2.2 (whether a change in knowledge is cardinal, ordinal or categorical).

Feature 5: Distribution of knowledge within simplices

KER might also explicitly include the distribution of knowledge in a simplex. In some cases (more applicable to learning than discovery), the rule may operate on the distribution of knowledge itself rather than explicitly on individual values per se. Various properties of distributions could be altered by rules, but for cardinal learning rules the four most common features are likely to be the mean, variance, skewness and kurtosis of the distribution of knowledge in a set. For example, a learning rule with a mixed trajectory (i.e., individuals can gain or lose knowledge) could reduce the kurtosis of distribution of knowledge values in a group, whereby the tails of the distributions became less heavy-tailed (fewer individuals have either very high or very low levels of knowledge). A key challenge for KER that operate on a distribution is re-assigning individual knowledge values, and specifications for how this is achieved may need to be included in a rule. One possible specification (for the example above) could be for existing and new knowledge values to be ranked and then paired according to rank. In this way individuals that initially had high knowledge will still be the more knowledgeable in the group and vice versa.

When applied to distribution means then it is possible to generate learning rules for which being in the group is sufficient to help an individual gain further knowledge (perhaps the most skillful individuals gain more skill from teaching others in the group), making it possible to model situations where simplex membership is mutually beneficial and knowledge levels need not converge. An alternative way to include properties of the distribution such as the variance, skewness or kurtosis in a KER would be in the term used to correct the gains of each individual for an individual-based learning rule. For example, individuals may gain more knowledge in a more variable group than a less variable one. This approach is simpler to apply than fully distribution-based rules but lacks some of the flexibility, meaning the preferred option will depend on the desired outcome.

Feature 6: Corrections for simplex size and membership

For both learning and discovery rules, changes in knowledge may additionally be governed by the order of the simplex and other properties of individuals in it.

Feature 6.1: Simplex size. Simplex size, which is the order of the simplex plus one (e.g. if it is a k order simplex, then it's size is $k+1$), might be an important inclusion in many rules. In some contexts individuals will not gain (or lose) as much when interacting in a large (high order) simplex than they would if the interacted in a smaller (lower-order) one, certainly when measured as a response per individual (e.g. [27]). In others changes in knowledge may be faster in larger groups (e.g. [24]). In situations where group size impacts knowledge gain or loss, including simplex order within the rule becomes important. For example, one simple way to include simplex order when individuals gain or lose knowledge more slowly in larger groups would be to divide all units of knowledge gained (in a simple learning rule) by the simplex size. However, other functional forms may be more appropriate depending on the situation being modeled.

In some contexts, individuals might be members of multiple simplices simultaneously. In these situations KER will have to a) determine in what order to handle the sets (e.g., in descending order of size or randomly) and b) may want to correct for membership of multiple simplices in each individual's change in knowledge (e.g., an individual gains less knowledge from each simplex it is a member of if it is part of three simultaneous sets than if it was a member of any one of those sets alone). More complex rules could even limit from which simplices an individual's change in knowledge is determined (e.g., an individual gains knowledge only from the largest simplex of which it is a member).

Feature 6.2: Simplex membership (additional traits). In most real-world situations groups will be heterogeneous. Thus, members will vary across multiple traits that might influence knowledge exchange (in token-passing algorithms) or the likelihood of joint discovery. Therefore, more complex rules might embrace this heterogeneity, for example, through weightings that account for high/low values of particular traits (e.g., perceived success) or similarity in these traits (e.g., higher weighting reflects closer agreement). Previous patterns of interactions may be a final consideration in considering heterogeneity in rules based on group membership. In simplicial sets with a temporal component, there could be weightings towards individuals that an individual has co-occurred with in a set previously (or indeed a bias in the opposite direction). Alternatively, external social data quantifying relationships among individuals in a population could be used instead. While heterogeneity will undoubtedly play a fascinating role in knowledge exchange on simplicial sets, it is beyond the scope of our current focus. We have provided some brief discussion here for completeness and future directions, however, our initial aim is to provide an overview of rules that treat individuals as homogeneous with respect to other traits.

Considerations when applying KER to simplicial sets

There are also a series of decisions to be made when applying knowledge exchange rules to simplicial sets, of which the principal ones are discussed here.

Influences of meta-structure on the application of KER in simplicial sets

For some simplicial sets, multiple simplices will be simultaneous, and in theory a single individual (zero simplex) could occur in more than one higher-order simplex at a given time. For example, imagine a scenario where someone is sat with a group of five friends (a five-simplex) but is talking on the phone to a family member (a one-simplex). In this scenario the individual is simultaneously participating in two simplices of different orders. When this is the case, the final distribution of knowledge can depend on the order that KER are applied to simplices. Various choices could be made when applying to structures such as these, with three obvious options being: random, increasing order and decreasing order. The optimal approach will depend on the biology of the system being studied.

Equivalently, there will also be decisions to be made about the order in which rules are applied to simplices of the same order (with a random order likely to be the most common choice). When individuals occur in multiple simplices simultaneously, then a further decision is whether to update their knowledge after applying rules to each of their simplices in turn, or only update their knowledge at the end of the time-step accounting for the knowledge they have gained from all of the simplices of which they are members. Taking this approach brings additional decisions about what this update should be (e.g. the maximum of the potential updates from each simplex or the aggregate for all of them), with the best option likely depending on the features of the knowledge exchange rule being used.

Stochastic applications of KER

KER with any of the features described above can also be applied stochastically. Stochasticity could be introduced in two main ways: in the implementation of the rule or to the rule itself. Starting with the former, determination of the (within time-step) process of applying the KER, or in the order of applying combination of rules, could be applied stochastically. For example, in contexts where rules were applied to simplices in increasing order (of size), this could be determined probabilistically, so that sometimes rules were applied to larger sets first. The second way stochasticity can be introduced is in the KER itself and this can also be done in two ways. The first is to add a stochastic component to the amount of knowledge an individual currently has. For example, a random variable could be added to the current value of someone's knowledge. Thus, an individual's current knowledge has a stochastic element regardless of the KER itself; adding stochasticity to the input. On the other hand, stochasticity can be added to the output. The latter would look like adding stochasticity as a component of the rule itself. For example, for learning rules, whether knowledge is gained could be probabilistic or the amount of knowledge gained could be drawn from a statistical distribution rather than being a fixed value. For discovery rules the probability of combining different types of knowledge could depend on the properties of the simplex (and of the individuals within it) rather than being a deterministic function of them. Depending on the amount of stochasticity desired, any combination of the methods described could be implemented.

Examples of knowledge exchange rules for simplicial sets

In this section we set out a non-comprehensive list of what we feel are likely to be the most commonly encountered learning and discovery rules.

Learning rules

We will start with some example learning rules. Borrowing and building on the notation from [14], let $c(X_j^k(v_i), t)$ be the learning rule acting on $v_i \in X_i^k$ at time t, $b_{v_i} \in \mathbb{R}^+$ be the knowledge value for $v_i \in X_j^k$ at time t, and m be the learning modifier. The learning modifier is a constant multiple that can be used to modify how much knowledge changes based on the rule in place. Note that $c(X_j^k(v_i), t)$ functions as part of an updating rule for $b_{v_i}(t+1)$. With this notation in mind, our first example of a simple learning rule is ***delta smartest***. This is a rule that applies in many educator/student type contexts, where knowledge gained by all individuals in a simplex is a function of their difference in knowledge to the smartest individual (i.e. the least knowledgeable gains the most from the interaction). Further corrections for simplex order can be included if desired. The delta smartest rule is defined as follows,

$$c(X_j^k(v_i), t) = m(\max\{b_{v_s}(t) \forall v_{s \neq i} \in X_j^k\} - b_{v_i}(t)). \tag{1}$$

The next learning rule we will consider it called ***delta all (general)***. Thus is an example of a learning rule in which the knowledge of individuals converges over time. The knowledge gained or lost by each individual is a function of the pairwise difference in knowledge to all other individuals in the group. In this first form the delta all rule is defined as,

$$c(X_j^k(v_i), t) = m\left(\sum_{\substack{v_s \in X_j^k \\ s \neq i}} b_{v_s}(t) - b_{v_i}(t)\right). \tag{2}$$

We can also define a second form of delta all, which we call ***delta all (simplex restricted)***. With this rule individuals can only increase their knowledge over time. Here we calculate the difference from all other individuals in the set and then only add this to an individual's knowledge if it is greater than zero (i.e. an individual will gain rather than lose knowledge). We write this second delta all rule as,

$$c(X_j^k(v_i), t) = m\left(\max\left\{0, \sum_{\substack{v_s \in X_j^k \\ s \neq i}} b_{v_s}(t) - b_{v_i}(t)\right\}\right). \tag{3}$$

To fully illustrate the simplicity of generating variation in these learning rules, we also define a third version of the delta all rule, called **delta all (dyad restricted)**. In this third form, we only add a difference from another individual when it is positive (i.e. that individual has greater knowledge than the focal individual). This can be defined as follows,

$$c(X_j^k(v_i), t) = m\left(\sum_{\substack{v_s \in X_j^k \\ s \neq i}} \max\left\{0, b_{v_s}(t) - b_{v_i}(t)\right\}\right). \tag{4}$$

Moving on from variations of the delta all rule, we have the ***delta average*** rule. This rule states all individuals change knowledge as a function of their difference in knowledge from the group average (selected as desired, in our case the mean). We define this rule as follows,

$$c(X_j^k(v_i), t) = m\left(\frac{\sum_{\substack{v_s \in X_j^k \\ s \neq i}} b_{v_s}(t)}{k+1}\right). \tag{5}$$

Similarly to the previous general rule (delta all), it is also possible to define alternative versions of the delta average rule. See S1 File for an example of a variation of the delta average rule.

In the ***gain fixed (basic)*** rule all individuals within a particular subset gain an equal unit of knowledge (subject to any corrections and weightings captured in Feature 6). How this subset is defined can be adapted to different purposes. In our example, we define a gain fixed rule whereby all individuals that co-occur in a set with Y more knowledgeable individuals gain F units of knowledge. We define the rule as follows,

$$c(X_j^k(v_i), t) = m\begin{cases} F, & \sum_{\substack{v_s \in X_j^k \\ b_{v_i}(t) < b_{v_s}(t)}} 1 > Y \\ 0, & \sum_{\substack{v_s \in X_j^k \\ b_{v_i}(t) < b_{v_s}(t)}} 1 \leq Y. \end{cases} \tag{6}$$

An easy adjustment of the gain fixed rule, ***the gain fixed (proportional)*** rule instead defines the subset of individuals that change their knowledge value using the proportion (Y) of other individuals in the simplex that have more/less knowledge. For example, under one such rule individuals could gain F knowledge if more than 50% of simplex members are more knowledgeable. This initial example of gain fixed (proportional) is defined as follows,

$$\begin{aligned} p &= \sum_{\substack{v_s \in X_j^k \\ b_{v_i}(t) < b_{v_s}(t)}} 1 \\ c(X_j^k(v_i), t) &= m\begin{cases} F, & \frac{p}{k} > Y \\ 0, & \frac{p}{k} \leq Y. \end{cases} \end{aligned} \tag{7}$$

This rule can facilitate incorporation of mixed trajectories (i.e. individuals can both gain and lose knowledge). For example, if we adjusted the definition of the rule to,

$$p = \sum_{\substack{v_s \in X_j^k \\ b_{v_i}(t) < b_{v_s}(t)}} 1$$
$$c(X_j^k(v_i), t) = m \begin{cases} F, & \frac{p}{k} > Y \\ -F, & \frac{p}{k} \leq Y. \end{cases} \tag{8}$$

This version of the rule allows individuals to lose knowledge if they are in the less knowledgeable half of the group. Altering the function further could allow different weightings to the gain and loss of knowledge and make it possible to consider the outcome of individuals gaining and losing knowledge at different rates in different higher-order structures. Just as with the other rules, more variations of the fixed gain rule are possible (see S1 File for another variation).

The last learning rule we will introduce is the ***dampening minimum*** rule. This rule reduces the sum of knowledge values over all participating individuals by a function of the difference between the most and least knowledgeable individual. This reflects scenarios in which the interaction rate is high among all participants, but all participants focus their efforts on teaching the least knowledgeable individual in the group. This can be expected when having low-knowledge participants may endanger the population (e.g., collaborative predator defence). We define the dampening minimum rule as follows,

$$c(X_j^k, t) = \frac{1}{k+1}\left\{\left(\sum_{v_i \in X_j^k} b_{v_i}(t)\right) - \left[m\Big(\max\{b_{v_i}(t)\forall v_i \in X_j^k\} - \min\{b_{v_i}(t)\forall v_i \in X_j^k\}\Big)\right]\right\}. \tag{9}$$

Table 1. Examples of learning rules. Evaluates features 2-5 for the example learning rules. ∗ indicated the rule can be found in S1 File.

Learning Rules				
	Feature 2	*Feature 3*	*Feature 4*	*Feature 5*
Delta Smartest	Cardinal knowledge and cardinal gain	Upward trajectory	Same function applied to all members	Individual-based rule
Delta All (General)	Cardinal knowledge and cardinal gain	Mixed trajectory	Same function applied to all members	Individual-based rule
Delta All (Simplex Restricted)	Cardinal knowledge and cardinal change	Upward trajectory	Same function applied to all members	Individual-based rule
Delta All (Dyad Restricted)	Cardinal knowledge and cardinal change	Upward trajectory	Same function applied to all members	Individual-based rule
Delta Average	Cardinal knowledge and cardinal change	Mixed trajectory	Same function applied to all members	Individual-based rule
Delta Average (Simplex Restricted)∗	Cardinal knowledge and cardinal gain	Upward trajectory	Same function applied to all members	Individual-based rule
Gain Fixed (Basic)	Cardinal or ordinal knowledge Cardinal or ordinal change	Upward trajectory	Different (sub-)functions applied to different subsets	Individual-based rule
Gain Fixed (Proportional)	Cardinal or ordinal knowledge Categorical change	Upward or mixed trajectory	Different (sub-)functions applied to different subsets	Individual-based rule
Gain Fixed (Per)∗	Cardinal or ordinal knowledge Categorical change	Upward or mixed trajectory	Different (sub-)functions applied to different subsets	Individual-based rule
Dampening Minimum	Cardinal knowledge and cardinal change	Mixed trajectory	Same function applied to all members	Individual-based rule

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Discovery rules

Before we can introduce some example discovery rules, we must first define the notation we will use. The discovery rules notation will rely very heavily upon mathematical set theory. Recall a set is simply a collection of distinct object or elements. Let $\mathbf{C}(X_j^k, t)$ be a finite set with M elements and let $K_m \in \mathbf{C}(X_j^k, t)$ be the mth element of that set. Then we write $\mathbf{C}(X_j^k, t) = \{K_m : 1 \leq m \leq M\}$, where $\mathbf{C}(X_j^k, t)$ represents the set of all knowledge for the simplex X_j^k at time t and $K_m \in \mathbf{C}(X_j^k, t)$ is an element of knowledge in that set. Next we want to define the set of knowledge for an individual vertex. Let $\mathbf{C}(X_j^k(v_i), t) \subseteq \mathbf{C}(X_j^k, t)$, where $\mathbf{C}(X_j^k(v_i), t)$ is the set of knowledge for $v_i \in X_j^k$ at time t. Then let $I = \{1, ..., M\}$ be the set of indices for the knowledge elements defined above and $I_{v_i} = \{m : K_m \in \mathbf{C}(X_j^k(v_i), t)\} \subseteq I$. We can write $\mathbf{C}(X_j^k(v_i), t) = \{K_m : \forall m \in I_{v_i}\}$. Note that an element $K_m \in \mathbf{C}(X_j^k, t)$ can be in multiple individual's sets of knowledge at once. For example let $v_i, v_s \in X_j^k$. Then $K_m \in \mathbf{C}(X_j^k(v_i), t), \mathbf{C}(X_j^k(v_s), t)$. Finally, we can also look at sets of vertices by the knowledge elements they have in their sets. Let $\mathbb{K}_m = \{v_s \in X_j^k : K_m \in \mathbf{C}(X_j^K(v_s), t)\}$. Thus $\mathbb{K}_m$ tell us the set of vertices have knowledge element K_m (which individuals "know" K_m). Note, if there is an interest in keep track of providence of the knowledge, see S1 File for a modification of the notation.

Recall that discovery rules capture situations in which knowledge is gained in a non-additive or cumulative manner. In its most basic form this means when individuals are found in a simplex with individuals that have different knowledge to them they will discover new knowledge together. Thus, each individual gains a new element of knowledge distinct from the elements of knowledge they had originally. Importantly, while each individual gains the new knowledge, they do not gain the individual elements that form the new, non-cumulative knowledge. We call this the ***discovery basic rule***. To describe this creation of new knowledge we introduce the concatenation operator, $+\!\!\!+$. We borrow this operator directly from formal notation of language theory; for any two elements of knowledge, K_m and K_n where $m \neq n$, then $K_m +\!\!\!+ K_n = K_m K_n$, a new element of knowledge that incorporates each component element, but is itself distinct and the novel combination arising from the two components. In this way, we capture that discovery hinges on learned knowledge, but produces a novel element. Next will define some properties for this new operator and describe how it interacts with elements of knowledge. (Note, for clarity we continue to employ use of the operator in our presentation.)

Let $K_m, K_p, K_n \in \mathbf{C}(X_j^k, t)$ and let m,p,n all be distinct. First we note that the concatenation operator is commutative, but not associative meaning that the outcome of combining the same pair of knowledge types is always the same but that once paired the properties change such that the order of combinations of knowledge types can matter. Or mathematically, $K_m +\!\!\!+ K_n = K_n +\!\!\!+ K_m$, but $(K_m +\!\!\!+ K_n) +\!\!\!+ K_p \neq K_m +\!\!\!+ (K_n +\!\!\!+ K_p)$. In this way, we allow for simultaneous multi-party discovery by concatenating all contributed elements within a collaboration and also for iterative discovery that builds on previous insights in a path-dependent way. This dynamic easily captures scenarios such as distributed decision-making algorithms employed in situational awareness and crisis management [8].

We also assume that concatenating the same knowledge has no effect. For example, $K_m +\!\!\!+ K_m = K_m$. Consider the following example of the discovery basic rule, which shows how the concatenation operator works with discovery knowledge exchange between two people:

> Let $v_1, v_2 \in X_j^1$. Further let individual, v_1, have knowledge $\mathbf{C}(X_j^1(v_1), t-1) = \{K_1\}$ and individual v_2 have knowledge $\mathbf{C}(X_j^1(v_2), t-1) = \{K_2\}$. Since v_1, v_2 co-occurs in simplex X_j^1, v_1 ends with knowledge $\mathbf{C}(X_j^k(v_1), t) = \{K_1, (K_1 +\!\!\!+ K_2)\}$ and v_2 ends with knowledge $\mathbf{C}(X_j^k(v_2), t) = \{K_2, (K_1 +\!\!\!+ K_2)\}$.

Note that while both v_1, v_2 have knowledge $K_1 +\!\!\!+ K_2$, v_1 does not have knowledge K_2 $\left(K_2 \notin \mathbf{C}(X_j^1(v_1), t)\right)$ and v_2 does not have knowledge K_1 $\left(K_1 \notin \mathbf{C}(X_j^1(v_2), t)\right)$. With this notation in place, we can formally define our discovery basic rule as follows,

$$\mathbf{C}(X_j^k(v_i), t) = \mathbf{C}(X_j^k(v_i), t-1) \bigcup_{\substack{\forall v_s \in X_j^k \\ s \neq i}} \{(K_m + K_n) : \forall K_m \in \mathbf{C}(X_j^k(v_i), t-1), \forall K_n \in \mathbf{C}(X_j^k(v_s), t-1)\}. \quad (10)$$

Now we can consider a slightly more complicated example of how the basic discovery rule works. Let $v_1, v_2 \in X_j^1$ and define the knowledge on the simplex at time t–1 as follows,

$$\begin{aligned} &\mathbf{C}(X_j^1, t-1) = \{K_1, K_2, K_3\} \\ &\mathbb{K}_1 = \{v_1, v_2\} \\ &\mathbb{K}_2 = \{v_1\} \\ &\mathbb{K}_3 = \{v_2\} \end{aligned} \quad (11)$$

With this information we can write out the knowledge sets of the vertices at t–1,

$$\begin{aligned} \mathbf{C}(X_j^1(v_1), t-1) &= \{K_1, K_2\} \\ \mathbf{C}(X_j^1(v_2), t-1) &= \{K_1, K_3\}. \end{aligned} \quad (12)$$

Then using the discovery basic rule above at time t we have,

$$\begin{aligned} \mathbf{C}(X_j^1(v_1), t) &= \{K_1, K_2\} \cup \{(K_1 + K_3), (K_1 + K_2), (K_2 + K_3)\} \\ &= \{K_1, K_2, (K_1 + K_3), (K_1 + K_2), (K_2 + K_3)\} \\ \mathbf{C}(X_j^1(v_2), t) &= \{K_1, K_3\} \cup \{(K_1 + K_2), (K_1 + K_3), (K_2 + K_3)\} \\ &= \{K_1, K_3, (K_1 + K_2), (K_1 + K_3), (K_2 + K_3)\}. \end{aligned} \quad (13)$$

Recall that a set is defined to have distinct elements, hence why there are no repeated elements in the knowledge sets at time t. In S1 File this example is worked again with the modified notation that tracks providence. Now that we have defined notation and formalized the basic discovery rule, we can expand on these ideas and explore other discovery rules. Note, along with the learning rules (summarized in Table 1), summarizing information about key features of the discovery rules presented below can be found in Table 2.

A similar rule to the discovery basic rule is the ***discovery per rule***. This rule allows individuals to discover new knowledge if found in a simplex with more than n (parameter to be varied) individuals that have particular knowledge. For example, let $n = 5$ and the desired knowledge element be K_2. Then an individual, v_i, with $\mathbf{C}(X_j^k(v_i), t-1) = \{K_1\}$, will have $\mathbf{C}(X_j^k(v_i), t) = \{K_1, K_1 + K_2\}$ if there are 6 or more individuals with knowledge K_2 $(|\mathbb{K}_2| \geq 6$, where $|\mathbb{K}_2|$ is the cardinality (number of elements) of the set $\mathbb{K}_2$). To define this rule mathematically let K_* be the desired knowledge element. Then we can write the discovery per rule as,

$$\mathbf{C}(X_j^K(v_i), t) = \begin{cases} \mathbf{C}(X_j^k(v_i), t-1), |\mathbb{K}_*| \leq n \\ \mathbf{C}(X_j^k(v_i), t-1) \cup \{(K_m + K_*) : \forall K_m \in \mathbf{C}(X_j^k(v_i), t-1)\}, |\mathbb{K}_*| > n. \end{cases} \quad (14)$$

Just like with the learning rules, there are many ways to make variations of these rules. Another example of a variation of the discovery basic rule can be found in S1 File.

Table 2. Examples of discovery rules. Evaluates features 2-5 for the example discovery rules. * indicated the rule can be found in S1 File.

Discovery Rules				
	Feature 2	*Feature 3*	*Feature 4*	*Feature 5*
Discovery Basic	Categorical knowledge and categorical change	Upward trajectory	Same function applied to all members	Individual-based rule
Discovery Per	Categorical knowledge and categorical change	Upward trajectory	Different (sub-)functions applied to different subsets	Individual-based rule
Discovery Proportional	Categorical knowledge and categorical change	Upward trajectory	Different (sub-)functions applied to different subsets	Individual-based rule
Interference Basic	Categorical knowledge and categorical change	Mixed trajectory	Same function applied to all members	Individual-based rule
Discovery and Learning (Basic)	Categorical knowledge and categorical change	Upward trajectory	Same function applied to all members	Individual-based rule

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While the first two discovery rules we introduced only allow for knowledge to increase or stay the same, there are contexts in which individuals may lose (or lose confidence in) their existing knowledge as they discover new information. This can be captured using an ***interference rule***. Our interference basic rule is defined so that if an individual v_i with $\mathbf{C}(X_j^k(v_i), t - 1) = \{K_1\}$ co-occurs in a simplex with an individual v_s with $\mathbf{C}(X_j^k(v_s), t-1) = \{K_2\}$ then the v_i ends with $\mathbf{C}(X_j^k(v_i), t) = \{K_1 + K_2\}$ and v_s ends with $\mathbf{C}(X_j^k(v_i), t) = \{K_1 + K_2\}$. It is formally defined as,

$$\mathbf{C}(X_j^k(v_i), t) = \bigcup_{\substack{\forall v_s \in X_j^k \\ s \neq i}} \{(K_m + K_n)^{v_i} : \forall K_m \in \mathbf{C}(X_j^k(v_i), t-1), \forall K_n \in \mathbf{C}(X_j^k(v_s), t-1))\} \tag{15}$$

Clearly it is possible to expand on interference rules in a whole number of ways, including the simple examples we have provided for the other discovery rules (e.g. requiring a minimum number or proportion of a group to possess particular knowledge for any change to occur).

Finally, we can consider a discovery rule that incorporates elements from learning rules. The ***discovery and learning (basic) rule*** allows individuals to acquire knowledge from each other (learning) as well as combining their existing knowledge to generate something new (discovery). Individuals learn from each other and discover new knowledge if found in a group with individuals that have different knowledge to them. For example, if an individual, v_i, with $\mathbf{C}(X_j^k(v_i), t-1) = \{K_1\}$ co-occurs with an individual v_s with $\mathbf{C}(X_j^k(v_s), t - 1) = \{K_2\}$ then the v_i ends with $\mathbf{C}(X_j^k(v_i), t) = \{K_1, K_2, K_1 + K_2\}$ and v_s ends with $\mathbf{C}(X_j^k(v_s), t) = \{K_2, K_1, (K_1 + K_2)\}$ The rule is defined as follows,

$$\mathbf{C}(X_j^k(v_i), t) = \bigcup_{\forall v_s \in X_j^k} \mathbf{C}(X_j^k(v_s), t-1) \bigcup_{\substack{\forall v_q \in X_j^k \\ q \neq i}} \{(K_m + K_n) : \forall K_m \in \mathbf{C}(X_j^k(v_i), t-1), \forall K_n \in \mathbf{C}(X_j^k(v_q), t-1))\}. \tag{16}$$

As mentioned previously, the learning and discovery rules provided here are examples of the types of rules that can be created to describe knowledge exchange on simplicial sets and is by no means an exhaustive list. Moving forward, more complex rules can be defined and specific situations described using the fundamentals set forth here.

Applying knowledge exchange rules

To illustrate how to apply some of these KERs in practice, we will use hypothetical, illustrative examples of seven KERs (four learning and three discovery) applied to a simple, hypothetical simplicial set (see Figs 5 and 6). Our hypothetical simplicial set is a simplified version of that used by [14] that was generated to represent the potential social structure of four interacting groups of New Caledonian crows. We retain the 3-simplex (v_0,v_1,v_2,v_3) and the three 2-simplices (v_1,v_4,v_5), (v_2,v_6,v_7) and (v_3,v_8,v_9), and consider knowledge exchange over a short time period (3 timesteps for learning and 2 for discovery).

We illustrate the delta smartest, delta all (simplex restricted), gain fixed (basic) and dampening minimum learning rules, parameterised as described in the figure (including choices about the order of application across simplices at each timestep during implementation). Within these example applications it is straightforward to see the differences in the positive trajectories of knowledge for the first three of these rules, as well as how quickly the dampening minimum rule could theoretically diminish variance in knowledge values across groups.

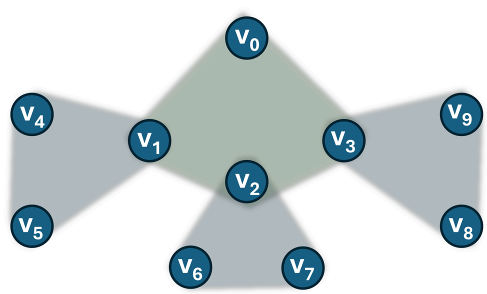
We also illustrate the discovery (basic), interference and discovery and learning rules to demonstrate differences in how they impact the distribution of new and existing knowledge across the simplicial set. Even from a single time step here, it is evident how quickly different forms of knowledge can combine in well-connected simplicial sets (especially when application of the rules is deterministic rather than probabilistic).

DELTA SMARTEST
m=0.5; *rule applied by increasing simplex size*

Vertex	t_0	t_1	t_2
v_0	10	10	10
v_1	2	6	8
v_2	4	7	8.5
v_3	5	7.5	8.75
v_4	2	2	4
v_5	2	2	4
v_6	1	2.5	4.75
v_7	2	3	5
v_8	2	3.5	5.5
v_9	1	3	5.25

DELTA ALL
m=0.2; simplex restricted; *rule applied by increasing simplex size*

Vertex	t_0	t_1	t_2
v_0	10	10	10
v_1	2	4.6	5.88
v_2	4	5	5.96
v_3	5	5.2	6
v_4	2	2	2.52
v_5	2	2	2.52
v_6	1	1.8	2.52
v_7	2	2.2	2.68
v_8	2	2.4	2.84
v_9	1	2	2.68



GAIN FIXED
Y=2; F=1; m=1; basic version; *rule applied by increasing simplex size*

Vertex	t_0	t_1	t_2
v_0	10	10	10
v_1	2	3	4
v_2	4	5	5
v_3	5	5	5
v_4	2	2	2
v_5	2	2	2
v_6	1	2	2
v_7	2	2	2
v_8	3	3	3
v_9	1	2	3

DAMPENING MINIMUM
m=0.2; *rule applied by decreasing simplex size*

Vertex	t_0	t_1	t_2
v_0	10	4.75	3
v_1	2	2.73	2.80
v_2	4	2.33	2.51
v_3	5	2.67	2.76
v_4	2	2.73	2.80
v_5	2	2.73	2.80
v_6	1	2.33	2.51
v_7	2	2.33	2.51
v_8	3	2.67	2.76
v_9	1	2.67	2.76

Fig 5. Example application of four learning rules to a simplified hypothetical simplicial set containing a 3-simplex and three 2-simplices.

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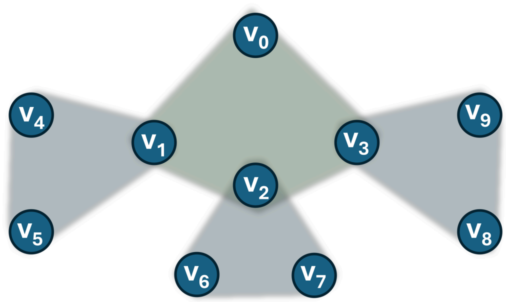
DISCOVERY BASIC

Vertex	t_0	T_1
v_0	A	A, AB
v_1	B	B, AB, BC
v_2	B	B, AB
v_3	B	B, AB, BD
v_4	C	C, BC
v_5	C	C, BC
v_6	A	A, AB
v_7	B	B
v_8	D	D, BD
v_9	D	D, BD

INTERFERENCE

Vertex	t_0	t_1
v_0	A	AB
v_1	B	AB, BC
v_2	B	AB
v_3	B	AB, BD
v_4	C	BC
v_5	C	BC
v_6	A	AB
v_7	B	B
v_8	D	BD
v_9	D	BD



DISCOVERY & LEARNING

Vertex	t_0	t_1
v_0	A	A, B, AB
v_1	B	A, B, C, AB, AC
v_2	B	A, B, AB
v_3	B	A, B, D, AB, AD
v_4	C	B, C, BC
v_5	C	B, C, BC
v_6	-	B
v_7	-	B
v_8	D	B, D, BD
v_9	D	B, D, BD

Fig 6. Example application of three discovery rules to a simplified hypothetical simplicial set containing a 3-simplex and three 2-simplices.

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Using our typology to design models across disciplines

Here we have combined a typology for KERs for higher-order networks with mathematical formulations for a non-exhaustive subset of potential examples (Tables 1 and 2) and an illustration of how these rules can be applied in practice. Clearly when designing models, whether it be for research field-specific theory or empirical research, the best choices will be closely determined by the specific research questions being addressed and key features of the system being addressed. For empirically-driven questions these choices will typically be informed or parameterised using real-world data.

Our typology can help researchers structure their thinking when designing models of knowledge exchange for higher-order networks by providing a clear route from desired outcomes or features to mathematical considerations. For applied researchers looking to decide on the best way to model their system then using our six features can help guide them towards a subset of the potential outcomes that will be most useful. The list of basic examples provided to illustrate how our typology works could then provide valuable starting points for researchers looking to develop their own models. Promoting effective cross-disciplinary research and minimising redundant progress across the diverse fields interested in these approaches can only be aided by a tool that provides a common framework and language around the development and application of models.

A logical next step to encourage further cross-disciplinary collaboration on this topic would be to set up an open access repository of these KER that is available freely and universally to researchers across multiple disciplines, and encoded using our typology. By providing a centralised, 'living' database in this way, researchers would be able to consider key features of the models they would like to design and use these features to identify past examples of similar models (regardless of research discipline).

Conclusions

Incorporating higher-order interaction structure has already been shown to be important to understanding social transmission and learning in varied contexts [2,6,16–19,25]. With this paper we have provided a route to classifying different rules for knowledge exchange within simplicial sets and hypergraphs to help researchers select and implement the most appropriate algorithms for their specific research questions. We feel the time right to promote interdisciplinarity in studying social contagion, information spread and knowledge exchange in complex social systems, and that a broad classification of KER, as we have provided here, is a crucial first step.

Supporting information

S1 File. The S1 File contains some additional examples of learning and discovery rules as well as alternative mathematical notation for implementing discovery rules.
(PDF)

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