Text S1. Stability analysis for steady states of approximating ordinary differential equation models.

Stability analysis for the ordinary differential equations system approximating the hemodynamics model

The only non-vanishing steady state of HD ordinary differential equations system is

\[ N^* = c_P. \]

Put \( u(N) := Nr \left( \frac{c_P}{N} \right) \) and observe that \( \frac{dN(t)}{dt} = u(N(t)) \) with

\[
u(N) = \begin{cases} 
NR_{\text{max}}, & N \leq \frac{c_P}{R_{\text{max}}/a-\hat{\tau}}, \\
-NR_{\text{min}}, & N \geq \frac{-c_P}{R_{\text{min}}/\alpha-\hat{\tau}}, \\
(c_P - N\hat{\tau})/a, & \text{otherwise.}
\end{cases}
\]

Evaluating the derivative of \( u \) at the steady state leads to

\[
\frac{du}{dN} \bigg|_{N=c_P} = -\hat{\tau} \alpha < 0,
\]

which means that the steady state is stable.

Stability analysis for the ordinary differential equations system approximating the metabolic load model

The only non-vanishing steady state of the ML ordinary differential equations system is

\[ (N^*, m^*, s^*) = (\hat{N}c_M, d, \hat{s}). \]

It is assumed that \( s_{\text{max}} \gg 0 \). Evaluating the Jacobian matrix at the steady state leads to

\[
J = \begin{pmatrix}
0 & 0 & c_M \hat{\beta} \\
-d & 0 & c_M \hat{\beta} \\
-Nc_M & -1 & 0 \\
0 & 1 & -1
\end{pmatrix}
\]

With the help of the Routh Hurwitz criterion \([1]\) statements about the stability can be made. Indeed, for the coefficients of the characteristic polynomial, it holds that

\[
c_0 = -\det(J) = \beta > 0 \\
c_1 = \det(J_{11}) + \det(J_{22}) + \det(J_{33}) = 1 > 0 \\
c_2 = -\text{spur}(J) = 2 > 0.
\]

In addition,

\[
c_1c_2 - c_0 = 2 - \beta > 0 \quad \text{for} \quad \beta < 2.
\]

Therefore, the steady state is stable for \( \beta < 2. \)

References