The objective function (Eq. (3)) is converted into a problem of minimizing pointwise a Hamiltonian \( H \) by Pontryagin’s Maximum Principle:

\[
H = I + \frac{c_1}{2} u_1^2 + \frac{c_2}{2} u_2^2 + \frac{c_3}{2} u_3^2 + \lambda_1 \{-\beta S(1 - \varepsilon_2 u_2(t))I + qA + \mu(NP_4 - S)\} \\
+ \lambda_2 \{\beta S(t)(1 - \varepsilon_2 u_2(t))I(t) + qA(t)) - kE(t) + \mu(N(t)P_4(t) - E(t))\} \\
+ \lambda_3 \{k(1 - \rho)E(t) - \gamma_1 A(t) + \mu(N(t)P_4(t) - A(t))\} \\
+ \lambda_4 \{k \rho E(t) - \gamma_2 I(t) - \varepsilon_3 u_3(t)I(t) + \mu(N(t)P_4(t)(1 - \varepsilon_3 u_3(t)) - I(t))\}
\]

(5)

According to this principle, if an optimal solution exists, then two conditions must be satisfied. First, continuous adjoint functions \( \lambda_i(t) \) must exist that satisfy

\[
\begin{align*}
\dot{\lambda}_1(t) &= -\frac{\partial H}{\partial S} = (\lambda_1 - \lambda_2) \beta((1 - \varepsilon_2 u_2(t))I + qA) + \lambda_4 \mu \\
\dot{\lambda}_2(t) &= -\frac{\partial H}{\partial E} = \lambda_2(k + \mu) - \lambda_3 k(1 - \rho) - \lambda_4 k \rho \\
\dot{\lambda}_3(t) &= -\frac{\partial H}{\partial A} = (\lambda_1 - \lambda_2) \beta Sq + \lambda_5 (\gamma_1 + \mu) \\
\dot{\lambda}_4(t) &= -\frac{\partial H}{\partial I} = \beta S(\lambda_1 - \lambda_2)(1 - \varepsilon_2 u_2(t)) + \lambda_4 (\gamma_2 + \varepsilon_1 u_3(t) + \mu) - 1
\end{align*}
\]

(6)

with transversality conditions \( \lambda_i(t_f) = 0, \quad i = 1, \ldots, 4 \). Second, the Hamiltonian \( H \) must be minimized with respect to the optimal control, satisfying the following optimality conditions:

\[
\begin{align*}
\frac{\partial H}{\partial u_1} &= c_1 u_1 - \lambda_4 \varepsilon_1 I = 0 \\
\frac{\partial H}{\partial u_2} &= c_2 u_2 + (\lambda_1 - \lambda_2) \beta S \varepsilon_2 I = 0 \\
\frac{\partial H}{\partial u_3} &= c_3 u_3 - \lambda_4 \mu NP_4 \varepsilon_1 = 0
\end{align*}
\]

(7)

Solving Eq. (7) for \( u_1, u_2, \) and \( u_3 \), we obtain:
\[
\begin{align*}
\dot{u}_1 &= \frac{\lambda_1 \varepsilon_1 I}{c_1}, \quad \dot{u}_2 = \frac{(\lambda_2 - \lambda_1) \beta S \varepsilon_2 I}{c_2}, \quad \dot{u}_3 = \frac{\lambda_3 \mu N P \varepsilon_3}{c_3} \\
\end{align*}
\] (8)

When this equation is combined with the restrictions of the control parameters, 
\[0 < u_i(t) < 1, \quad i = 1, 2, 3, \quad t \in (t_0, t_f)\] (adjusted according to the specific assumptions), the control parameters of the optimal solution are:

\[
\begin{align*}
\dot{u}_1^* &= \min\{\max(0, \frac{\lambda_1 \varepsilon_1 I}{c_1}), 1\} \\
\dot{u}_2^* &= \min\{\max(0, \frac{(\lambda_2 - \lambda_1) \beta S \varepsilon_2 I}{c_2}), 1\} \\
\dot{u}_3^* &= \min\{\max(0, \frac{\lambda_3 \mu N P \varepsilon_3}{c_3}), 1\}
\end{align*}
\] (9)

The integrand in Eq. (3) is a convex function of \((u_1, u_2, u_3)\), and the state equation in Eq. (1) satisfies the Lipshitz property. The existence and uniqueness of the optimal controls can be proven according to the literature [1,2].

Given the initial value of the state equation according to the actual situation, the problem of solving the optimal solution of the system converts into a two-point boundary value problem. Because Eqs. (3) and (6) are coupled nonlinear differential equations, it is difficult to find analytic solutions. Using a fourth-order Runge-Kutta method, we created an iterative program to simulate numerical solutions for \(u_1^*(t)\), \(u_2^*(t)\), and \(u_3^*(t)\) that satisfied the restrictions. Finally, the antiviral resource consumption can be calculated according to:

\[
M = \int_{t_0}^{t_f} u_i(t) I(t) dt
\] (10)

References
