Appendix S3: Second-Order Correction to the Posterior Expectation

Here we will derive the second-order correction to the expectation of the posterior distribution \([Y_t|\xi_t, Y_{\leq t-1}]\). We have:

\[
E(Y_t|\xi_t, Y_{\leq t-1}) = E_{X_t|\xi_t}\left\{ E(Y_t|X_t, Y_{\leq t-1}) \right\} \\
\approx E_{X_t|\xi_t}\left\{ E(Y_t|X_t = \xi_t, Y_{\leq t-1}) + (X_t - \xi_t)^T \left[ \frac{\partial}{\partial X_t} E(Y_t|X_t, Y_{\leq t-1}) \right]_{X_t = \xi_t} \right. \\
+ \frac{1}{2} (X_t - \xi_t)^T \left[ \frac{\partial^2}{\partial X_t^2} E(Y_t|X_t, Y_{\leq t-1}) \right]_{X_t = \xi_t} \left( X_t - \xi_t \right) \right\} \\
= E(Y_t|X_t = \xi_t, Y_{\leq t-1}) + \frac{1}{2} \text{trace} \left\{ V(X_t|\xi_t) \left[ \frac{\partial^2}{\partial X_t^2} E(Y_t|X_t, Y_{\leq t-1}) \right]_{X_t = \xi_t} \right\}. \tag{1}
\]