Appendix S2: Stability of Equilibria

Here we consider the stability of equilibrium solutions to Main Text Eqs. 11-13 for the case \( u(t) \) a constant with \( u = v \). To simplify then notation we set \( s = k(p + 1) \) and \( r = a(1 - u)/u \) (note we have excluded \( u = 0 \)) and rescale all time constants in units of time that correspond to \( uk\delta\rho k = 1 \). In this case the equations reduce to:

\[
\frac{dx}{dt} = \frac{1}{s + e^{\gamma x}} - \mu + \alpha q, \tag{19}
\]
\[
\frac{dq}{dt} = -\frac{rq}{s + e^{\gamma x}} - c\mu - b\alpha q. \tag{20}
\]

First we note that since \( s > 0, r > 0, b > 0, c > 0, k > 0, \alpha > 0, \mu > 0 \) and \( \gamma \geq 1 \), it follows that \( q(t) \) is bounded above by 0 (since \( \frac{dq}{dt}|_{q=0} = -c\mu < 0 \)). Thus we are interested in equilibrium solutions \((\hat{x}, \hat{q})\) for which \( \hat{q} \leq 0 \) and the corresponding equilibrium total mortality rate on \( x \) is

\[
\hat{m} = \mu - \alpha \hat{q} > 0, \tag{21}
\]

from which it follows that an equilibrium solution \( \hat{x} \) exists and is given by

\[
\frac{1}{s + e^{\gamma \hat{x}}} = \mu - \alpha \hat{q} \implies e^{\gamma \hat{x}} = \frac{1}{\mu - \alpha \hat{q}} - s \implies \hat{x} = \frac{1}{\gamma} \ln \left( \frac{1}{\mu - \alpha \hat{q}} - s \right) \text{ provided } \mu - \alpha \hat{q} > \frac{1}{s}. \tag{22}
\]

If we now set the rhs of Eq. 20 to zero and substitute the first expression in Eq. 22, we obtain the quadratic equation in \( \hat{q} \)

\[
r\alpha \hat{q}^2 - (r\mu + b\alpha)\hat{q} - c\mu = 0 \tag{23}
\]

that has the two solutions

\[
\hat{q}_\pm = \frac{(\alpha b + \mu r) \pm \sqrt{4\alpha\mu cr + (\alpha b + \mu r)^2}}{2\alpha r}. \tag{24}
\]

Since all parameters are positive, \( q_+ > 0 \) and \( q_- < 0 \). Thus the latter is the only one applicable to our analysis and below when we use the notation \( \hat{q} \) then \( \hat{q}_- \) is implied and \( \hat{x} \) is the corresponding value of \( x \) obtain using Eq. 22.

Linearizing Eqs. 19 and 20 around an applicable equilibrium \((\hat{x}, \hat{q})\), we obtain the Jacobian stability matrix

\[
J(x, q) = \begin{pmatrix}
\frac{-\gamma e^{\gamma x}}{(s + e^{\gamma x})^2} & \frac{\alpha}{s + e^{\gamma x}} \\
\frac{-\gamma e^{\gamma x} q}{(s + e^{\gamma x})^2} & \frac{-r}{s + e^{\gamma x}} - b\alpha
\end{pmatrix}. \tag{25}
\]

Noting from Eqs. 21 and 22 that

\[
\hat{q} = \frac{\mu - \hat{m}}{\alpha} \quad \text{and} \quad \frac{e^{\gamma \hat{x}}}{(s + e^{\gamma \hat{x}})^2} = \hat{m} \left( 1 - s\hat{m} \right)
\]

with a solution existing provided

\[
s\hat{m} - 1 > 0 \tag{26}
\]
it follows that

\[ J(\hat{x}, \hat{q}) = \begin{pmatrix} \gamma \hat{m} (s \hat{m} - 1) & \alpha \\ \gamma r (\hat{m} - \mu) \hat{m} (s \hat{m} - 1) / \alpha & -r \hat{m} - b \alpha \end{pmatrix} \]

The eigenvalues of \( \hat{J} \) are

\[ \hat{\lambda}_\pm = \frac{\text{tr} \hat{J} \pm \sqrt{\left(\text{tr} \hat{J}\right)^2 - 4 \det \hat{J}}}{2} \]

where

\[ \text{tr} \hat{J} = -b\alpha - (\gamma + r)\hat{m} + \gamma s \hat{m}^2. \]

Since all the parameters are positive it follows from Eqs. 21 (recalling \( \hat{q} < 0 \)) and 26 that

\[ \det \hat{J} = \gamma (s \hat{m} - 1)(ab + r \hat{m} + r(\hat{m} - \mu)) > 0. \]

Thus the eigenvalues are real and at least one is negative. Thus the equilibrium \((\hat{x}, \hat{q})\) is either a stable node or saddle point, thereby excluding local oscillations in arbitrary small neighborhoods of the solution \((\hat{x}, \hat{q})\).