Supplementary material

Mathematical analysis of dynamics and response times

Socialist motif

With the socialist motif, for the up-shift perturbations we start $\sigma = 1$. In this steady state, $T$ is large, while $E$ and $s$ are small: $T \approx 1, E = 0.2, s = 0.05$. Initially, just after the perturbation, when $\sigma$ is changed to $\sigma' = \sigma + \Delta \sigma$, $E$ and $T$ remain constant at their initial level. During this time, $s$ evolves according to the equation

$$
\frac{ds}{dt} = \sigma'T - (\gamma E + 1)s = \sigma' - 20s
$$

$\Rightarrow$ $s(t) = s(0)e^{-20t} + \frac{\sigma'}{20}(1 - e^{-20t})$

In other words, $s$ starts from the initial value of $s(0) = 0.05$ and exponentially relaxes towards the target value of $s_f = \sigma'/20$ at a rate 20. Thus, the peak height of the overshoot is $\sigma'/20$ and the time taken to reach within 95% of this level is $3/20 \approx 0.15$ (to get this we have assumed that the peak overshoot height is much larger than the starting level of $s$). Note that the overshoot height is proportional to the size of the perturbation, while the time is independent of it.

The final steady state level of $s$ for the socialist motif grows slower than linearly with $\sigma$, i.e. $s(\infty) \sim (\sigma')^\alpha$ where $\alpha < 1$. The initial response time, $t_1$, can be calculated from the previously derived equation

$\Rightarrow$ $s(t) = s(0)e^{-20t} + \frac{\sigma'}{20}(1 - e^{-20t})$

Essentially, for very small times only the first term on the right hand side is important

$\Rightarrow e^{-20t} \approx s(t) / s(0)$

$\Rightarrow t_1 \sim 1 / \ln(\sigma')$

i.e. $t_1$ decreases, but slower than linearly, as $\Delta \sigma$ is increased.

The second response time, $t_2$, is dominated by the time required for $s$ to fall from the peak overshoot $\sigma'/20$ to the final $s(\infty) \sim (\sigma')^\alpha$, which is much smaller. Therefore, as a first approximation $t_2 \sim \ln(\sigma')$ (the larger the perturbation, the better this approximation). That is, $t_2$ grows as the logarithm of the perturbation size $\Delta \sigma$.

Overall, for the socialist motif subjected to up-shifts in $\sigma$, we have shown above that:

i) the peak overshoot height is $\sigma'/20$,
ii) the time of peak overshoot is independent of $\sigma'$ and is $\approx 0.15$,
iii) the initial response time decreases, slower than linearly, as $\Delta \sigma$ is increased,
iv) the second response time increases, slower than linearly, as $\Delta \sigma$ is increased.

This explains all the observations made in the main text about Figs. 1 and 2.
A similar analysis can be done for downshifts. We start with $\sigma = 10^4$, at which steady-state $s \approx 3\, E \approx 1\, T \approx 0.03$. Initially, just after the perturbation, when $\sigma$ is changed to $\sigma' = \sigma + \Delta\sigma$, $E$ and $T$ remain constant at their initial level. During this time, $s$ evolves according to the equation

$$\frac{ds}{dt} = \sigma'T - (\gamma E + 1)s \approx 0.03\sigma' - 100s$$

$$\Rightarrow s(t) = s(0)e^{-100t} + \frac{3}{10^4}(1 - e^{-100t})$$

In other words, $s$ starts from the initial value of $s(0) = 3$ and exponentially relaxes towards the target value of $s_f = 3\sigma'/10^4$. Thus, the $s$ level at peak overshoot is $3\sigma'/10^4$.

For sufficiently large perturbations ($\sigma \leq 1000$), this is much lower than the initial $s$ level (the converse is true for up-shifts) the last term on the right hand side above is negligible, and the peak overshoot time $\sim$ const. $\ln(\sigma)$. That is the peak overshoot time increases, slower than linearly, as the perturbation size $\Delta\sigma$ is increased.

As for upshifts, the initial response time, $t_1$, can be calculated from the previously derived equation

$$\Rightarrow s(t) = 3e^{-100t} + \frac{3}{10^4}(1 - e^{-100t})$$

For very small times only the first term on the right hand side is important

$$\Rightarrow e^{-100t} = s(t) / r$$

$$\Rightarrow t_1 = 1 / \ln(\sigma')$$

i.e. $t_1$ increases, but slower than linearly, as $\Delta\sigma$ is increased.

The second response time, $t_2$, is dominated by the time required for $s$ to increase from the peak overshoot $3\sigma'/10^4$ to the final steady state $s(\infty) \approx (\sigma')^n$, which is much larger. For large enough perturbations, this is dominated by the timescale of one cell generation, which is required for $E$ and $T$ to reach their final steady states. Over this main timescale, $t_2$ also has contributions from $t_1$ and the peak overshoot time; therefore it also grows as $\Delta\sigma$ is increased, but again slower than linearly.

Overall, for the socialist motif subjected to up-shifts in $\sigma$, we have shown above that:

i) the peak overshoot height is $3\sigma/10^4$,
ii) the time of peak overshoot grows slower than linearly as $\Delta\sigma$ is increased,
iii) the initial response time increases, slower than linearly, as $\Delta\sigma$ is increased,
iv) the second response time also increases, slower than linearly, as $\Delta\sigma$ is increased.

This explains all the observations made in the main text about Figs. 1 and 2.
**Consumer motif**

The analysis of the consumer motif is easier. For upshifts, we start with $\sigma = 1$, and $s \approx 0.005$, $T = E \approx 0.01$. In this case the positive feedback is off. As soon as $\Delta \sigma$ is larger than around 1, the perturbation is big enough to start switching on the positive feedback. Already for $\Delta \sigma = 10$ the positive feedback is almost completely on. To get to this on state both $E$ and $T$ must increase, which happens on a timescale of one cell generation. Thus, the response of the consumer to up-shifts is of that timescale.

For downshifts, we start with $\sigma = 10^4$. At this value, $s \approx 100$, $T = E \approx 1$, i.e. the positive feedback is switched on. $\Delta \sigma$ has to be quite large (roughly $> 9990$) for the perturbation to be large enough to start switching the feedback off. For $\Delta \sigma < 9990$ the feedback remains on, with $T = E \approx 1$. In this regime, the $s$ dynamics is given by

\[
\frac{ds}{dt} = \sigma' - 100s
\]

\[
\Rightarrow s(t) = s(0)e^{-100t} + 0.01\sigma'(1 - e^{-100t})
\]

Thus, for small enough perturbations the response time is $\approx 3/100 = 0.03$. For larger perturbations, where the feedback is switched off, and $E$ and $T$ change significantly, the response time as usual becomes of the order of one cell generation.

Thus, as seen in Figs. 1 and 2 of the main text,

i) for up-shifts the consumer motifs responds on the timescale of one cell generation,

ii) for small enough down-shifts the response time is of the order of one hundredth of a cell generation,

iii) for very large down-shifts, the response is again of the order of one cell generation.