Local Linearisation

If the deterministic part of the dynamics evolve according to a linear differential equation

\[ \dot{x} = Ax + Bu \]  

then a discrete time update is given by

\[ x(t) = \exp(At)x(0) + \int_0^t \exp(A(t-\tau))Bu(\tau)d\tau \]  

For time step \( n \), if we assume that \( u(t) = 0 \) except at \( t = t(n) \) then we have

\[ x_n = \exp(Adt)x_{n-1} + Bu_n \]  

where \( dt \) is the time step. If \( u_n \) is not changing quickly we have \( u_n = u_{n-1} \). For nonlinear dynamics

\[ \dot{x} = f(x, u) \]  

then we can write

\[ x_n = F_n x_{n-1} + H_n u_{n-1} \]  

where the flow matrices are given by

\[ F_n = \exp(Jf(x, x)dt) \]
\[ H_n = J(f, v)dt \]  

and \( J(f, x) \) is the Jacobian matrix of the function \( f \) with respect to \( x \) (matrix of first derivatives). In forward inference, these are evaluated at \( x = m_{n-1} \) and \( u = u_{n-1} \) (for known causes) or \( u = r_{n-1} \) (for estimated causes).

However, our evaluations of the above approximations for \( F_n \) and \( H_n \) showed considerable inaccuracies for a range of angles, \( \phi \). We therefore adopted the following ‘local regression’ approach which is similar to that proposed by Schaal [28]. This used multiple, typically 10, expansion points sampled from the previous posterior \( (m_{n-1}, P_{n-1}) \). For each, we evaluated the gradient \( f(x, u) \) and estimated the next state based on a first order Euler method. We then regressed the next states onto previous states and computed \( F_n \) and \( H_n \) using least squares regression.