Text S1  Analysis of quenched oscillator system for satisfying the three conditions for Turing instability

To obtain parameter constraints for Turing instability, we first linearize the reaction terms in (6)-(16) at the steady state \((\bar{m}_C, \bar{p}_C, \bar{m}_{TO}, \bar{p}_T, \bar{m}_L, \bar{p}_L, \bar{m}_I, \bar{p}_I, \bar{A}, \bar{p}_{RA}, \bar{m}_{TQ})\) and obtain the Jacobian matrix:

\[
J = \begin{bmatrix}
-\gamma_m O & 0 & 0 & -b_2 & 0 & 0 & 0 & 0 & 0 & 0 \\
c_C & -\gamma_C & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -\gamma_m O & 0 & 0 & -b_0 & 0 & 0 & 0 & 0 \\
0 & 0 & \epsilon_{TO} & -\gamma_T & 0 & 0 & 0 & 0 & 0 & \epsilon_{TQ} \\
0 & -b_2 & 0 & 0 & -\gamma_m O & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \epsilon_L & -\gamma_L & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -b_{12} & 0 & 0 & -\gamma_{mQ} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \epsilon_I & -\gamma_I & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & v_3 & -a_9 & a_{10} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_9 & -a_{10} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & b_{10} & -\gamma_{mQ}
\end{bmatrix},
\]

where we use the parameters \(\alpha_C = \bar{p}_C/K_C, \alpha_T = \bar{p}_T/K_T, \alpha_L = \bar{p}_L/K_L, \) and \(\alpha_A = \bar{A}/K_A\) to obtain the off-diagonal entries:

\[
c_9 = \frac{k_{pr}}{1 + \alpha_A}, \quad a_9 = c_9 + \gamma_A, \quad a_{10} = k_r (1 + \alpha_A), \\
b_2 = V_{Pr} V_T C_{\alpha C}^{nC_{\alpha C}} \frac{N_C}{K_C (1 + \alpha_C)^{n_C}}, \\
b_4 = V_{P_{LatO}} N_C C_{\alpha C}^{nC_{\alpha C}} \frac{N_C}{K_C (1 + \alpha_C)^{n_C}}, \\
b_42 = V_{P_{LatO}} N_C C_{\alpha C}^{nC_{\alpha C}} \frac{N_C}{K_C (1 + \alpha_C)^{n_C}}, \\
b_6 = V_{P_{LatO}} N_C T C_{\alpha C}^{nC_{\alpha C}} \frac{N_C}{K_C (1 + \alpha_C)^{n_C}}, \\
b_{10} = V_{P_{LatO}} N_C T C_{\alpha C}^{nC_{\alpha C}} \frac{N_C}{K_C (1 + \alpha_C)^{n_C}}.
\]

\(J_{osc}\) is the 6 \(\times\) 6 principal submatrix of \(J\), which corresponds to the first loop – the standard repressilator system (\(cI-lacI-tetR\)). AHL is the only diffusible species, so \(D = \text{diag} \{0, 0, 0, 0, 0, 0, d_{AHL}, 0, 0\}\).

**Condition 1:** The oscillator loop by itself would produce oscillations (\(J_{osc}\) is unstable).

The eigenvalues of \(J_{osc}\) are the roots of:

\[
\text{det}(\lambda I - J_{osc}) = (\lambda + \gamma_{mO})^3 (\lambda + \gamma_p)^3 + \epsilon_C \epsilon_{TO} \epsilon_L b_2 b_4 b_6.
\]  

(S.1)

It can be shown [s9] that instability of \(J_{osc}\) is achieved when:

\[
\frac{(\beta + 1)^2}{\beta} < \frac{3X^2}{4 + 2X}
\]  

(S.2)

where \(\beta = \gamma_p/\gamma_{mO}\) and \(X = -\frac{1}{\gamma_p \gamma_{mO}} \sqrt{\epsilon_C \epsilon_{TO} \epsilon_L b_2 b_4 b_6}\). Substituting steady-state expressions and rearranging, we arrive at the following expression for \(X\):

\[
X^3 = -n_C n_T n_L \frac{\alpha_{RA} \alpha_{C}^{nC} \alpha_{T}^{nT}}{1 + \alpha_{RA} + \alpha_{C}^{nC} + \alpha_{T}^{nT} + \ell_{Pr}(1 + \alpha_{C}^{nC}) + \ell_{T}(1 + \alpha_{T}^{nT}) + \ell_{P_{LatO}} (1 + \alpha_{T}^{nT})} \\
\times \frac{1}{1 + \frac{\alpha_{RA}^{nRA}}{\alpha_{RA}^{nRA}}} \\
\times \frac{1}{1 + \frac{\alpha_{L}^{nL}}{\alpha_{L}^{nL}}} \\
\times \frac{1}{1 + \frac{\alpha_{L}^{nL}}{\alpha_{L}^{nL}}}
\]

(S.3)

where the additional variable \(\alpha_{RA} \geq 0\) is defined by the relation:

\[
\frac{1}{1 + \alpha_{RA}} = \frac{\epsilon_{TQ} V_{P_{LatO}} N_C T Q C}{\gamma_p \gamma_{mO} \alpha_T K_T} \left(1 + \frac{K_{EA} (1 + \alpha_A)^{n_{RA}} + \ell_{P_{LatO}}}{\alpha_{RA}}\right).
\]
**Condition 2:** The quenching loop ceases oscillations in the full system ($J$ is stable).

The eigenvalues of $J$ are the roots of:

$$\det(\lambda I - J) = \det(\lambda I - J_{osc})(\lambda + \gamma_I)(\lambda + \gamma_{mO})^2[(\lambda + a_9)(\lambda + a_{10}) - c_9a_{10}] + F(\lambda + \gamma_{mO})^3(\lambda + \gamma_p)^2,$$

where $F = v_{34}v_{TQ}c_{9}b_{42}b_{10}$ characterizes the feedback strength. To quench the oscillatory modes of $J_{osc}$, $F$ must be a value such that all of the eigenvalues of $J$ have negative real part. Substituting steady-state expressions and rearranging, we arrive at the following expression for $F$:

$$F = \gamma_I^2\gamma_A^2\gamma_{mQ}^2k_Tn_{RA}^2\frac{1}{1 + \alpha_{RA}^2}\frac{(K_{RA}^2 + \alpha_{A}^2)n_{RA}^2}{1 + \alpha_{RA}^2}\frac{1}{1 + \ell_{Plux}^2(1 + (K_{RA}^2 + \alpha_{A}^2)n_{RA}^2)}$$

$$(S.5)$$

**Condition 3:** Diffusion will weaken the quenching loop’s influence on the oscillator loop for high wave numbers, allowing spatio-temporal oscillations to emerge ($J + \lambda_kD$ is unstable for some $k \geq 1$).

For $\Omega = [0, L]$, $\lambda_k = -(k\pi/L)^2$ for eigenfunctions $\cos(k\pi x)$. $J + \lambda_kD$ looks identical to $J$ except for the AHL entry of the diagonal, which is now defined as $-\hat{a}_9 = -c_9 - \gamma_A + \lambda_kd_{AHL}$. This leads to:

$$\det(\lambda I - (J + \lambda_kD)) = \det(\lambda I - J_{osc})(\lambda + \gamma_I)(\lambda + \gamma_{mQ})^2[(\lambda + \hat{a}_9)(\lambda + a_{10}) - c_9a_{10}] + F(\lambda + \gamma_{mO})^3(\lambda + \gamma_p)^2,$$

which yields unstable roots for large enough $\lambda_kd_{AHL}$. This implies that, for diffusion-driven patterning, we need a large diffusion coefficient, a large wave number, or a small spatial domain. Let $d_{thresh}$ be the instability threshold for a particular set of parameters, that is, (S.6) becomes unstable when:

$$(k\pi/L)^2d_{AHL} > d_{thresh}.$$  

This expression can be rewritten in terms of the spatial wavelength $\omega_x$ as:

$$\omega_x^2 < 4\pi^2d_{AHL}/d_{thresh}.$$  

This maximum unstable wavelength is a convenient formulation because it applies to any chosen spatial domain size.

**References**