The dynamics of T-cell receptor repertoire diversity following thymus transplantation for DiGeorge Anomaly- Text S1

Stanca M. Ciupe*, Blythe H. Devlin†, M. Louise Markert†‡, Thomas B. Kepler⋆‡

Because the number of lineages is large and because we study the long-term average behavior of $x_i$ we can replace the stochastic terms by their expectation per unit time to study the average or steady-state behavior. At steady-state, all clones have the same size. Indeed, if we consider that thymic emigration has attained its maximal value $\epsilon_M = \epsilon \tau \eta$, we can derive the steady state solution for a family $x_i$ to be

$$x_i = \frac{\gamma - \frac{\gamma(1-\rho)}{\epsilon_M}}{\frac{\gamma(1-\rho)}{\epsilon_M} + \frac{\sqrt{\left(\frac{\gamma - \frac{\gamma(1-\rho)}{\epsilon_M}}{\epsilon_M}\right)^2 + 4\epsilon_M \frac{\gamma(1-\rho)}{\epsilon_M}}}{2}},$$

(1)

which is independent of specificities $i$. Therefore at equilibrium all clone sizes are identical and equal to $x_i = \frac{T}{n}$. The total number of T cells satisfies the equation

$$n\epsilon_M + \gamma T - \frac{\gamma}{\kappa} T^2 = 0.$$

(2)

Moreover, if the thymic contribution to the peripheral pool is negligible at steady state, $\epsilon_M \ll \gamma T$, the total number of T cells at equilibrium is

$$T = \kappa.$$

(3)

We can show that the steady state $(\frac{T}{n}, \frac{T}{n}, ... , \frac{T}{n})$ is asymptotically stable. We linearize Eqs.(1,4) about the steady state (1) to get

$$A = \begin{pmatrix}
\gamma - \frac{\gamma (2n-1)\rho + 1}{n} & 0 & \cdots & 0 \\
-\frac{\gamma (1-\rho)}{n} & \gamma - \frac{\gamma (2n-1)\rho + 1}{n} & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots \\
-\frac{\gamma (1-\rho)}{n} & \gamma - \frac{\gamma (2n-1)\rho + 1}{n} & \cdots & \gamma - \frac{\gamma (1-\rho)}{n}
\end{pmatrix}$$

(4)

The characteristic equation for the linearized system is

$$(\gamma - \frac{2\gamma}{\kappa} T - \Lambda) (\gamma - \frac{\gamma (1 + \rho)}{\kappa} T - \Lambda)^{n-1} = 0.$$

(5)

From equation (2) we have that $\gamma - \frac{2\gamma}{\kappa} T = -n\epsilon_M / T$. We can conclude that all eigenvalues of (5),

$$\Lambda_1 = -\frac{\gamma}{\kappa} T - \frac{n\epsilon_M}{T},$$

$$\Lambda_2, \ldots, n = -\frac{\gamma (1 + \rho)}{\kappa} T - \frac{n\epsilon_M}{T},$$

(6)

are negative, therefore the steady state $(\frac{T}{n}, \frac{T}{n}, ..., \frac{T}{n})$ is asymptotically stable.