The calculation of volume changes over time

This section describes a method for calculating the time evolution of an initial volume in the phase space. We used this method to determine the extent of variance reduction in single-nucleus gene circuits (Fig. 4C,D).

Given the diffusionless gap gene dynamical system in the nucleus at A–P position \( x \) (Eq. 2)

\[
\frac{dv^a}{dt} = R^a g \left( \sum_{b=1}^{N} T^{ab} v^b + m^a v^Bcd(x) + \sum_{\beta=1}^{N_c} E^{a\beta} v^\beta(x,t) + h^a \right) - \lambda^a v^a,
\]

we want to calculate the volume \( V_t \) at time \( t \) of a box of initial conditions \( B_0 \). Let us represent the right hand side as \( f(v_t, t) \), where \( v_t = (v_1^t, v_2^t, v_3^t, v_4^t) \) is the four-dimensional state vector at time \( t \).

The ODE is discretized into a map,

\[
v_{t+dt} = \gamma(v_t, t), \tag{S1}
\]

where \( \gamma(v_t, t) = v_t + f(v_t, t)dt \). Regard the map at time \( t + (n-1)dt \),

\[
v_{t+ndt} = \gamma(v_{t+(n-1)dt}, t + (n-1)dt),
\]

as a curvilinear coordinate transformation of the phase space, then the infinitesimal volume at time \( t + ndt \),

\[
dV_{t+ndt} = dv^1_{t+ndt} dv^2_{t+ndt} dv^3_{t+ndt} dv^4_{t+ndt}
\]

can be written as,

\[
dV_{t+ndt} = J(v_{t+(n-1)dt}, t + (n-1)dt) dV_{t+(n-1)dt}. \tag{S2}
\]

Here, \( dV_{t+(n-1)dt} \) is the infinitesimal volume at \( t + (n-1)dt \), and \( J(v_t, t) \) is the Jacobian of the map at time \( t \). Applying Eq. (S2) repeatedly, the infinitesimal volume at \( t, dV_t \), can be written in terms of the initial infinitesimal volume, \( dV_0 \) as

\[
dV_t = J(v_{t-(n-1)dt}, t - (n-1)dt) J(v_{t-(n-2)dt}, t - (n-2)dt) \ldots J(v_0, 0) dV_0.
\]

If the initial box \( B_0 \) evolves to \( B_t \) at time \( t \), \( B_t \)'s volume is

\[
V_t = \iiint_{B_t} dV_t
= \iiint_{B_0} [J(v_{t-(n-1)dt}, t - (n-1)dt) \ldots J(v_0, 0)] dv^1_0 dv^2_0 dv^3_0 dv^4_0.
\]
The integral on the right hand side was evaluated using the multidimensional trapezoidal rule [1], successively refining the grid on the initial box $B_0$ until the integral converged. The time-step for the discretized map $\gamma(v_t, t)$ was chosen small enough such that the Euler-method solution of Eq. (2) converged.

References