Supporting Text S1

Model neurons

We use the Wang-Buzski (WB) conductance-based model (ref. [86] in main text) to describe each single excitatory and inhibitory neuron. The WB model is described by a single compartment endowed with sodium and potassium currents. The membrane potential is given by:

\[ C \frac{dV}{dt} = -I_L - I_{Na} - I_K + I_{ext} + I_{rec} \]

where \( C \) is the capacitance of the neuron, \( I_L = g_L(V - V_L) \) is the leakage current, \( I_{ext} \) is an external driving current and \( I_{rec} \) is due to recurrent interactions with other neurons in the network (see later). Sodium and potassium currents are voltage-dependent and given by \( I_{Na} = g_{Na}m_\infty h(V - V_{Na}) \) and \( I_K = g_Kn_\infty(V - V_K) \). The activation of the sodium current is instantaneous:

\[ m_\infty(V) = \frac{\alpha_m(V)}{\alpha_m(V) + \beta_m(V)} \]

Sodium current inactivation and potassium current activation evolve according to:

\[ \frac{dx}{dt} = \Phi \cdot (\alpha_x(V)(1 - x) - \beta_x(V)x) \]

where \( x = h, n \) and \( \alpha_x \) and \( \beta_x(V) \) are non-linear functions of the membrane potential given by:

\[
\begin{align*}
\alpha_m(V) &= 0.1(V + 35) / (1 + e^{-\frac{V+35}{10}}) \\
\beta_m(V) &= 4e^{-\frac{V+60}{15}} \\
\alpha_n(V) &= 0.03(V + 34) / (1 - e^{-\frac{V+34}{10}}) \\
\beta_n(V) &= 0.375e^{-\frac{V+44}{80}} \\
\alpha_h(V) &= 0.21e^{-\frac{V+58}{20}} \\
\beta_h(V) &= \frac{3}{1 + e^{-\frac{V+28}{10}}} 
\end{align*}
\]

Other parameters are \( g_{Na} = 35 \text{ mS/cm}^2 \), \( V_{Na} = 55 \text{ mV} \), \( g_K = 9 \text{ ms/cm}^2 \), \( V_K = -90 \text{ mV} \), \( g_L = 0.1 \text{ mS/cm}^2 \), \( C = 1 \mu \text{F/cm}^2 \) and \( \phi = 5 \).

Model synapses

The synaptic current induced in a postsynaptic neuron by a single presynaptic action potential is given by \( I_{spike}(t) = -g_x s_{spike}(V - V_x) \), where \( V \) is the potential in the postsynaptic neuron and \( V_x \) is the reversal potential of the synapse (for excitatory synapses \( V_E = 0 \text{ mV} \), for inhibitory synapses \( V_I = -80 \text{ mV} \)). The time-course of the postsynaptic conductance is described by:

\[ s_{spike}(t) \propto (\exp(-(t + d - t^*)/\tau_1) - \exp(-(t + d - t^*)/\tau_2)) \]
for $t > t^*$, 0 otherwise, where $t^*$ is the time of the presynaptic spike, $d$ is the latency, $\tau_1$ the rise-time and $\tau_2$ the decay-time. The total recurrent current $I_{\text{rec}}(t)$ is the sum of time-dependent contributions $I_{\text{spike}}(t)$ from all the presynaptic spikes fired to time $t$. The normalization constant of $s_{\text{spike}}(t)$ is chosen such as the peak value of $s_{\text{spike}}$ is equal to 1. For all simulations in the paper, we take $\tau_1 = 1$ ms, $\tau_2 = 3$ ms and $d = 0.5$ ms. Thus, post-synaptic currents have a relatively fast decay, corresponding to AMPA-like excitatory and GABA$_A$-like inhibitory synapses. For simplicity, we take only two possible peak conductances, $g_I = 90 \mu S/cm^2$ for inhibitory synapses within an area and $g_E = 5 \mu S/cm^2$ for excitatory synapses within and between areas.

**Parameters of the background noise**

In addition to recurrent synaptic inputs, each neuron receives a noisy input, representing background spiking activity. It is modeled as an excitatory current having the same functional form of a recurrent current induced by a Poisson spike train with firing rate $f_{\text{ext}}$. The peak conductance of this noisy background input is $g_{\text{ext}}$. In our simulations, we take $f_{\text{ext}} = 5$ kHz, and $g_{\text{ext}} = g_E = 5 \mu S/cm^2$. Each neuron is driven by statistically independent Poisson noise realizations.

**Phase response of the rate model**

As previously throughly reported in the Supplementary Material of ref. [42] (in main text), the firing rate of a single oscillating area (only local inhibitory coupling $K_I < 0$ with delay $D$) can be derived analytically assuming that: (i) the total input current $I_{\text{tot}}(t) = I + K_IR(t - D)$ is below threshold (i.e. negative) for a duration $T_{st} > D$; (ii) the delay $D$ and the oscillation period $T$ fulfill the inequalities $D < T - T_{st} < 2D$. The conditions (i) and (ii) hold for sufficiently strong local inhibition, and, specifically, for the value $K_I = -250$ and the delay $D = 0.1$ adopted in the main paper. Under these conditions, the limit cycle of the firing rate assumes then the following analytic form (see Figure 2B in the main paper):
The term $\Gamma(\Delta \phi)$ is a functional of the phase response and of the limit cycle waveform of the uncoupled oscillating areas. In terms of the previously derived analytic expressions of $Z(\phi)$ and of the rate oscillation limit cycle $R(\phi)$ (phase-reduced) for $K_E = 0$, this functional can be expressed as $\Gamma(\Delta \phi) = C(\Delta \phi) - C(-\Delta \phi)$, where:

$$C(\Delta \phi) = \int_0^1 Z(\phi)R(\phi + \Delta \phi - D)d\phi$$
Stable phase-lockings are therefore given by the zeroes of $\Gamma$ with negative slope crossing. Analytic expressions for the integral $C(\Delta \phi)$ have already been derived and published in the Supplementary Material of ref. [31] (in main text). We report here these expression again, in order to make the presentation of results self-contained. To compute $C(\Delta \phi)$, six different intervals of $\Delta \phi$ need to be considered separately. The result is:

$$
C(\Delta \phi) = \begin{cases} 
C_{00}(\phi_{st}, 1) + C_{10}(\phi_{st}, 1 - \phi_D) & \Delta \phi \in [\phi_D - \phi_{st}, \phi_{st} + \phi_D - 1] \\
C_{00}(\phi_{st}, 1) + C_{10}(\phi_{st}, 1 - \phi_D) + C_{01}(\phi_{st} + \phi_D - \Delta \phi, 1) & \Delta \phi \in [\phi_{st} + \phi_D - 1, \phi_{st} + 2\phi_D - 1] \\
C_{00}(\phi_{st}, 1) + C_{10}(\phi_{st}, 1 - \phi_D) + C_{01}(\phi_{st} + \phi_D - \Delta \phi, 1) + C_{11}(\phi_{st} + \phi_D - \Delta \phi, 1 - \phi_D) + C_{02}(\phi_{st} + 2\phi_D - \Delta \phi, 1) & \Delta \phi \in [\phi_{st} + 2\phi_D - 1, \phi_D] \\
C_{00}(\phi_{st}, 1 + \phi_D - \Delta \phi) + e^TC_{00}(\phi_{st} + \phi_D - \Delta \phi, 1) + C_{10}(\phi_{st}, 1 - \phi_D) + C_{11}(\phi_{st} + \phi_D - \Delta \phi, 1 - \phi_D) + C_{12}(\phi_{st} + 2\phi_D - \Delta \phi, 1 - \phi_D) & \Delta \phi \in [\phi_D, \phi_{st} - 1 + 3\phi_D] \\
C_{00}(\phi_{st}, 1 + \phi_D - \Delta \phi) + e^TC_{00}(\phi_{st} + \phi_D - \Delta \phi, 1) + C_{10}(\phi_{st}, 1 - \phi_D) + C_{01}(\phi_{st} + \phi_D - \Delta \phi, 1 + \phi_D - \Delta \phi) + C_{11}(\phi_{st} + \phi_D - \Delta \phi, 1 + \phi_D - \Delta \phi) + C_{12}(\phi_{st} + 2\phi_D - \Delta \phi, 1 - \phi_D) & \Delta \phi \in [\phi_{st} - 1 + 3\phi_D, 2\phi_D] \\
C_{00}(\phi_{st}, 1 + \phi_D - \Delta \phi) + e^TC_{00}(\phi_{st} + \phi_D - \Delta \phi, 1) + C_{10}(\phi_{st}, 1 + \phi_D - \Delta \phi) + C_{11}(\phi_{st}, 1 + \phi_D - \Delta \phi) + C_{12}(\phi_{st}, 1 + \phi_D - \Delta \phi) & \Delta \phi > 2\phi_D
\end{cases}
$$
where

\[ C_{00}(a, b) = -(b - a)Te^{-T(1 - \Delta \phi + \phi_D)} \]

\[ C_{10}(a, b) = Ke^{T(2\phi_D - 1 - \Delta \phi)} \left[ \frac{T(x + \phi_D - 1)^2}{2} \right]^b_a \]

\[ C_{01}(a, b) = -K_2e^{D - T} \left[ T(b - a)e^{T(\phi_D - \Delta \phi)} - e^{-T}\left(e^{bT} - e^{aT}\right) + e^{T(\phi_D - \Delta \phi)} \left[ \frac{T(x + \Delta \phi - \phi_D - \phi_{st})^2}{2} \right]^b_a \right] + e^{T(\phi_D - \Delta \phi)} \left[ \frac{T(x + \Delta \phi - 2\phi_D - \phi_{st})^2}{2} + \frac{T(x + \Delta \phi - 2\phi_D - \phi_{st})^3}{6} \right]^b_a \]

\[ C_{11}(a, b) = K^2e^{2D - T} \left[ e^{T(\phi_D - \Delta \phi)} \left( \frac{(bT)^2}{2} - \frac{(aT)^2}{2} + T(D - T)(b - a) \right) - e^{-T}\left((x - 1)e^{xT} + (D - T)e^{xT}\right)_a + e^{T(\phi_D - \Delta \phi)} \left[ \frac{1}{3}(xT + D - T)^3 + T(x + \Delta \phi - 2\phi_D - \phi_{st})(xT + D - T)^2 \right]^b_a \right] \]

\[ C_{12}(a, b) = K^3e^{3D - T} \left[ e^{T(\phi_D - \Delta \phi)} \left( \frac{(bT)^2}{2} - \frac{(aT)^2}{2} + T(D - T)(b - a) \right) - e^{-D - T}\left((x - 1)e^{xT} + (D - T)e^{xT}\right)_a + e^{T(\phi_D - \Delta \phi)} \left[ \frac{1}{3}(xT + D - T)^3 + T(1 + \Delta \phi - 3\phi_D - \phi_{st})(xT + D - T)^2 + \frac{(xT + D - T)^4}{8} + T(1 + \Delta \phi - 3\phi_D - \phi_{st})^2(xT + D - T)^2 \right]^b_a \right] \]

where \([f(x)]_a^b = f(b) - f(a)\). A plot of \(\Gamma(\Delta \phi)\) for the parameters used in our study is reported in Figure 4B (main text).